Optimal low-thrust limited-power transfers in a non-central gravity field - transfers between arbitrary orbits

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Abstract. This paper presents a study of optimal low-thrust limited-power trajectories in a non-central gravity field which includes the oblateness of the Earth (i.e., includes the second zonal harmonic $J_2$ in the gravitational potential). Initially, the space trajectories optimization problem is expressed in Mayer form, with Cartesian elements - position and velocity vectors - as state variables. The concepts of Optimum Control Theory, namely, the Pontryagin Maximum Principle, are then applied to this problem, providing a maximum Hamiltonian. The resulting formulation is afterwards modified by introducing a set of suitable orbital elements through a canonical transformation. A generalized canonical version of Hori method (a perturbation technique based on Lie series) is then applied to this new formulation, resulting in an averaged maximum Hamiltonian, which characterizes long duration transfers. A numerical solution for this problem is obtained for some transfers between arbitrary elliptical orbits, employing an algorithm based on the second variation method (neighboring extremals) to solve the two-point boundary value problem associated with the averaged Hamiltonian. The results obtained are compared to those provided by the same method for the undisturbed problem (central gravity field), in order to analyze the influence of $J_2$ in the transfers considered.

1 Introduction

In the study of the dynamics of aerospace vehicles, such as satellites and probes, one topic of great interest has been the determination of trajectories, named optimal, along which the mission accomplishment leads to the best possible value for a certain parameter, usually called performance index or cost index. These missions, either simple transfers or rendez-vous, are especially important in Space Mechanics and the problem of determining such trajectories is called Space Trajectories Optimization Problem.
Such problems have been the subject of several analytical and numerical studies since the early 1950’s, with the establishment of the fundamentals of the modern space navigation through the classical Calculus of Variations [1, 2]. Later, the enunciation of Pontryagin’s Maximum Principle [3, 4, 5] allowed a new methodology to study the optimization problems, which has been used on space trajectories.

In these studies, the main analytical results have been developed for small amplitude transfers with fixed duration, performed through low-thrust limited-power (LP) propulsive systems, and for impulsive transfers between arbitrary orbits with free duration [6, 7]. With respect to numerical studies, results have been obtained by employing techniques strongly related to the optimization theories. Both direct methods, such as the first order gradient [8], and indirect methods, such as the second variation [9], have been used.

The greater part of the above mentioned studies, either analytical or numerical, performed in the 1960’s and 1970’s and part of the 1980’s, consider the hypothesis of central Newtonian gravity field. The possibility of considering secondary forces was pointed out by Marchal [10] and has attracted the interest of researchers. In this scenario, studies involving the more general hypothesis of non-central gravity field, considering small amplitude transfers performed by LP systems, have been developed using the methods of perturbations theory [11, 12, 13, 14]. More general maneuvers have also been considered [15, 16, 17]. This paper, motivated by the these works, presents a study of optimal low-thrust limited-power trajectories in a non-central gravity field.

This hypothesis of non-central field is introduced by the simple consideration, in the problem formulation, of a different mathematical model for the gravitational potential. More specifically, the second zonal harmonic \( J_2 \) (i.e., the Earth oblateness effect) is included. This consideration introduces significant complications in the analysis of the problem. However, the existence of a canonical structure allows to employ perturbation methods, especially those based on Lie series [18, 19].

In order to do this, a generalized version of Hori method [20] is used, resulting in an averaged maximum Hamiltonian, which characterizes long duration transfers. A numerical solution for this problem is obtained for some transfers between arbitrary elliptical orbits, employing an algorithm based on the second variation method. Results obtained are compared to those provided by the same method for the undisturbed problem (central field), in order to analyze the influence of \( J_2 \).

2 Optimal space transfers

2.1 Space trajectories optimization problem

Consider a space vehicle \( M \) subject to the gravitational attraction and the application of a propulsive force \( F \). Its movement in an inertial reference frame is described by [21]:

\[
m \frac{d^2 r}{dt^2} = mg(r,t) + F,
\]

where \( m(t) \) is the instantaneous mass of the vehicle, \( r \) is the radius vector (position), \( g \) is the gravitational acceleration and \( F \) is the propulsive force resulting from mass ejection. In the problems considered herein, \( F \) is assumed to be freely oriented. Restrictions over its magnitude, however, depend on the propulsion system.

An optimal space transfer is considered as the movement of a space vehicle from an initial orbit \( O_0 \) to a final orbit \( O_f \), performed through the application of the propulsive force \( F \) and with the smallest possible fuel consumption. This minimization criterion is certainly not the only possible one, but it is
the most used for practical purposes. The transfer duration may or may not be fixed. If the position of
the vehicle on the final orbit is prescribed, the maneuver is called rendez-vous, otherwise, it is called
simple transfer or, merely, transfer [21, 22].

In order to treat the optimization problem described above, it is necessary to know the propulsive
force $F$ and the gravitational acceleration $g$ as a function of time or express them as functions of the
variables of the problem. In the latter case, which is the most interesting for practical problems, the
mathematical modeling depends on the propulsive system used, while the modeling of $g$ depends on
the gravitational field considered.

Regarding the force $F$, the classical low-thrust limited-power (LP) system [21] is considered in
this work. For this system, the following quantity is usually adopted as a performance index:

$$\frac{dJ}{dt} = \frac{1}{2} \gamma^2,$$

which relates to $F$ by: $F/m = \gamma D$, where $\gamma$ is the magnitude of the acceleration provided by the
propulsive force and $D$ is its direction (unit vector).

The quantity $J$ in Eq. (2.2) is a decreasing monotonic function of the mass of the vehicle. Con-
sequently, minimizing its final values, $J_f$, corresponds to minimizing the fuel consumption for the
associated transfer problem. This is the criterion adopted herein.

On the other hand, the gravitational potential of an ellipsoid of revolution, considering only the
oblateness effect, is given by [23]:

$$U = -\frac{\mu}{r} \left[ 1 - \frac{a_e^2}{r^2} J_2 P_2 (\sin \phi) \right],$$

where $\mu$ is the gravitational parameter, $a_e$ is the mean equatorial radius, $r$ is the distance from a point
P in the space to the center of mass of the ellipsoid, $\phi$ is the latitude of point P, $J_2$ is the coefficient
related to the second zonal harmonic ($J_2 = 1.08263 \times 10^{-3}$) and $P_2$ is the Legendre polynomial of 2nd
order. For the potential in Eq. (2.3), the gravitational acceleration $g$ is given by:

$$g(r,t) = -\nabla U = -\frac{\mu}{r^3} r + \left( \frac{\partial V_2}{\partial r} \right)^T,$$

where $V_2$ is the disturbing force function,

$$V_2 = -\mu a_e^2 \frac{J_2}{r^3} P_2 (\sin \phi).$$

### 2.2 Optimal control problem

Combining Eqs. (2.1) and (2.2), the space trajectories optimization problem formulated in the previ-
sous section can be transcribed to the classical formulation of optimal control, using the Mayer form
[24, 25, 26, 27]. In this way, using vectors position and velocity ($r$ and $v$), besides the quantity $J$, the
optimal control problem in study consists in transferring the system defined by:

$$\frac{dr}{dt} = v$$
$$\frac{dv}{dt} = g(r,t) + \gamma D$$
$$\frac{dJ}{dt} = \frac{1}{2} \gamma^2,$$
from the initial state (orbit): \( O_0 : (r_0, v_0, 0) \), at initial time \( t_0 \), to the final state: \( O_f : (r_f, v_f, J_f) \), at prescribed final time \( t_f \), in such a way as to minimize the performance index:

\[ IP = J_f. \]  

(2.7)

### 2.3 Optimal control law

In accordance with Pontryagin Maximum Principle, one introduces the adjoint variables \( \lambda_r \), \( \lambda_v \) and \( \lambda_J \) and writes the Hamiltonian function \( H \) in the form:

\[ H = \lambda_r v + \lambda_v g(r, t) + \lambda_J \gamma D + \frac{1}{2} \gamma^2 \lambda_J = H_0 + \lambda_v \gamma D + \frac{1}{2} \gamma^2 \lambda_J, \]  

where \( H_0 \) is independent of the control. The operator "\( \cdot \)" denotes the scalar vector product. Introducing into Eq. (2.8) the expression for the non-central gravity field in Eq. (2.4) and applying the Pontryagin Maximum Principle [4], results that the maximum Hamiltonian is given by:

\[ H^* = H_0^* + H_J^* + H_\gamma^*, \]  

(2.9)

where:

\[ H_0^* = \lambda_r v - \lambda_v \frac{\mu}{r^3} r, \]  

(2.10)

\[ H_J^* = \lambda_v \left( \frac{\partial V_2}{\partial r} \right)^T, \]  

(2.11)

\[ H_\gamma^* = \frac{1}{2} \lambda_r^2, \]  

(2.12)

and, the optimal thrust acceleration is given by:

\[ \gamma^* = \lambda_v, \]  

(2.13)

that is, it is modulated [21].

In the above equations, \( H_0^* \) denotes the undisturbed Hamiltonian, which describes the movement in a central Newtonian field, and \( H_J^* \) and \( H_\gamma^* \) are disturbing functions, assumed to be of the same order of magnitude.

### 3 Canonical transformation

In order to analyze the dynamics of the optimum transfer characterized by the maximum Hamiltonian in Eq. (2.9), a perturbation technique, namely, the Hori method, is employed. First, however, it is convenient to write Eq. (2.9) in terms of orbital elements, through a canonical transformation, using concepts of generalized canonical systems [28].

#### 3.1 Generalized canonical systems

Consider a Hamiltonian function \( \mathcal{H} \), associated to a canonical system with an integrable part:

\[ \mathcal{H}(x, \lambda) = H(x, \lambda) + R(x, \lambda) = \lambda \cdot f(x) + \lambda \cdot g(x), \]  

(3.1)
where $H$ denotes the integrable part, $R$ is the disturbing function, $x, \lambda \in \mathbb{R}^n$ and the functions $f_i(\cdot), g_i(\cdot), i = 1, \ldots, n$ and the partial derivatives of $R$ with respect to $x$ and $\lambda$ are assumed to satisfy the hypotheses of the Existence and Uniqueness Theorem.

The integrable part of $\mathcal{H}$ describes the following dynamical system:

\[
\begin{align*}
\frac{dx}{dt} &= f(x), \quad (3.2) \\
\frac{d\lambda}{dt} &= -\left[ \frac{\partial f}{\partial x} \right]^T \lambda, \quad (3.3)
\end{align*}
\]

whose solution, assumed periodic, is given by [28]:

\[
\begin{align*}
x &= \psi(\theta_1, \ldots, \theta_m, c_{m+1}, \ldots, c_n), \quad (3.4) \\
\lambda &= (\Delta_\psi^{-1})^T b, \quad (3.5)
\end{align*}
\]

where $\theta_i, i = 1, \ldots, m$ are $m$ fast phases, $c_j, j = m+1, \ldots, n$ are $n$ arbitrary constants of integration, $b$ is a vector of implicit time functions through the arbitrary phases and other $n$ integration constants and $\Delta_\psi$ is the Jacobian matrix:

\[
\Delta_\psi = \begin{bmatrix}
\frac{\partial \psi}{\partial \theta_1} & \cdots & \frac{\partial \psi}{\partial \theta_m} & \frac{\partial \psi}{\partial c_{m+1}} & \cdots & \frac{\partial \psi}{\partial c_n}
\end{bmatrix}.
\quad (3.6)
\]

It can easily be proved [28] that the general solution given by Eqs. (3.4) and (3.5) defines a canonical transformation – a Mathieu transformation:

\[
(\lambda; x) \xrightarrow{\text{Mathieu}} (b_1, \ldots, b_n; \theta_1, \ldots, \theta_m, c_{m+1}, \ldots, c_n)
\]

The Hamiltonian function $\mathcal{H}$ in (3.1) is invariant with respect to this transformation:

\[
\mathcal{H}(x, \lambda) = \mathcal{H}^*(\theta_1, \ldots, \theta_m, c_{m+1}, \ldots, c_n, b_1, \ldots, b_n) = b^T \Delta_\psi^{-1} (f^*(c) + g^*(c)), \quad (3.7)
\]

where:

\[
\begin{align*}
c &= [\theta_1 \ldots \theta_m \ c_{m+1} \ldots c_n]^T, \\
f^*(c) &= f(x(\theta_1, \ldots, \theta_m, c_{m+1}, \ldots, c_n)), \quad (3.8) \\
g^*(c) &= g(x(\theta_1, \ldots, \theta_m, c_{m+1}, \ldots, c_n)). \quad (3.9)
\end{align*}
\]

The new dynamical system governed by the new Hamiltonian $\mathcal{H}^*$ is autonomous and is defined by:

\[
\begin{align*}
\frac{dc}{dt} &= \omega + \Delta_\psi^{-1} g^*(c), \quad (3.11) \\
\frac{db}{dt} &= -\left[ \frac{\partial}{\partial c} \left( \Delta_\psi^{-1} g^*(c) \right) \right]^T b, \quad (3.12)
\end{align*}
\]

where $\omega = [\omega_1 \ldots \omega_m 0 \ldots 0]^T$, being $\omega_i, i = 1, \ldots, m$, frequencies of the the system, which are constants in the undisturbed case.
3.2 Transformation from cartesian to orbital elements

The general results presented above for generalized canonical systems are used to transform $H^*$ in Eq. (2.9) from Cartesian elements (position and velocity vectors) to classical orbital elements (Keplerian).

Consider the dynamical system associated to the undisturbed Hamiltonian $H_0^*$ in Eq. (2.10):

$$\frac{dr}{dt} = v$$
$$\frac{dv}{dt} = -\frac{\mu}{r^3} r$$
$$\frac{d\lambda_r}{dt} = \frac{\mu}{r^3} (\lambda_r - 3 (\lambda_r, r) \hat{r})$$
$$\frac{d\lambda_v}{dt} = -\lambda_r,$$

(3.13)

where $\hat{r}$ denotes the unit vector in the radial direction. Similar definitions apply for $\hat{s}$ and $\hat{w}$: $\hat{s}$ is the unit vector along circunferential direction and $\hat{w}$ is the unit vector normal to the plane of the orbit.

The state equations in Eq. (3.13) correspond to the dynamics of the 2-body problem. Their general solution is well known [29] and given by:

$$r = a (1 - e \cos E) \hat{r} = \frac{a (1 - e^2)}{1 + e \cos v} \hat{r},$$
$$v = \sqrt{\frac{\mu}{a (1 - e^2)}} [e \sin v \hat{r} + (1 + e \cos v) \hat{s}],$$

(3.14) (3.15)

with

$$\hat{r} = [\cos \Omega \cos (\omega + v) - \sin \Omega \sin (\omega + v) \cos I] \hat{i} +$$
$$+ [\sin \Omega \cos (\omega + v) + \cos \Omega \sin (\omega + v) \cos I] \hat{j} +$$
$$+ [\sin (\omega + v) \sin I] \hat{k},$$
$$\hat{s} = -[\cos \Omega \sin (\omega + v) + \sin \Omega \cos (\omega + v) \cos I] \hat{i} +$$
$$+ [-\sin \Omega \sin (\omega + v) + \cos \Omega \cos (\omega + v) \cos I] \hat{j} +$$
$$+ [\cos (\omega + v) \sin I] \hat{k},$$
$$\hat{w} = \sin \Omega \sin I \hat{i} - \cos \Omega \sin I \hat{j} + \cos I \hat{k},$$

(3.16)

where $\hat{i}, \hat{j}, \hat{k}$ denote the unit vectors in the intertial reference frame. The solution in Eqs. (3.14) and (3.15) involves 5 arbitrary integration constants - $a, e, I, \Omega, \omega$ - and one fast phase - $M$. They constitute the classical orbital elements - $a$ is the semimajor axis, $e$ is the eccentricity, $I$ is the the inclination of the plane of the orbit, $\Omega$ is the longitude of ascending node, $\omega$ is the argument of perihelion and $M$ is the mean anomaly. Additionally, Eqs. (3.14) and (3.15) involve the true anomaly $v$ and the eccentric anomaly $E$. They are related to $M$ through well-known expressions of the two-body problem [29].

From Eqs. (3.14) and (3.15) and in view of the last section, one can define:

$$c = [a \ e \ I \ \Omega \ \omega \ M]^T,$$
$$b = \pi = [\pi_a \ \pi_e \ \pi_I \ \pi_{\Omega} \ \pi_\omega \ \pi_M]^T,$$
$$\varpi = [0 \ 0 \ 0 \ 0 \ n]^T.$$

(3.17) (3.18) (3.19)
where $n$ is the mean motion: $n = \sqrt{\mu/a^3}$. In order to obtain the general solution for the adjoint equations, the next step is to calculate the inverse Jacobian Matrix in Eq. (3.5), as presented above. The following inverse transformation property is used in this process:

$$\Delta_{\psi^{-1}} = \Delta_{\psi}^{-1}. \quad (3.20)$$

Performing the calculations, the following expressions for the adjoint vectors are obtained [30]:

$$\lambda_{rr} = \frac{a}{r^2} \left\{ 2a\pi_\omega + (1 - e^2) \cos E \pi_e + \frac{1}{e} \left( \frac{r}{a} \right) \sin \nu \pi_\omega - \frac{(1 - e^2 \cos E)}{e\sqrt{1 - e^2}} \left( \frac{r}{a} \right) \sin \nu \pi_M \right\} \hat{r}$$

$$+ \left\{ \frac{\sin \nu}{a} \pi_e - \frac{(\cos \nu + e)}{ae(1 - e^2)} \pi_\omega + \frac{\sqrt{1 - e^2}}{ae} \cos \nu \pi_M \right\} \hat{s} + \frac{1}{a\sqrt{1 - e^2}} \left( \frac{a}{r} \right) \times$$

$$\left\{ \sin E \left[ \cos \omega \pi_I + \sin \omega \left( \frac{\pi_\Omega}{\sin I} - \cot I \pi_\omega \right) \right] + \sqrt{1 - e^2} \cos E \left[ \sin \omega \pi_I - \cos \omega \left( \frac{\pi_\Omega}{\sin I} - \cot I \pi_\omega \right) \right] \right\} \hat{w}, \quad (3.21)$$

$$\lambda_v = \frac{1}{na\sqrt{1 - e^2}} \left\{ \left\{ 2ae \sin \nu \pi_\omega + (1 - e^2) \sin \nu \pi_e - \frac{(1 - e^2)}{e} \cos \nu \pi_\omega + \frac{(1 - e^2)^{3/2}}{e} \left( \frac{a}{1 + e \cos \nu} + \cos \nu \right) \pi_M \right\} \hat{r} + \left\{ 2a(1 - e^2) \left( \frac{a}{r} \right) \pi_\omega + (1 - e^2) \right\} \hat{s} +$$

$$\left\{ \left( \frac{a}{r} \right) \cos (\omega + \nu) \pi_I + \left( \frac{a}{r} \right) \sin (\omega + \nu) \left( \frac{\pi_\Omega}{\sin I} - \cot I \pi_\omega \right) \right\} \hat{w} \right\}, \quad (3.22)$$

where $(a/r)$, $(r/a) \sin \nu$, $(r/a) \cos \nu$, etc. are well-known expressions of the elliptical motion, functions of mean anomaly and eccentricity [23].

Finally, in view of the equations above and Eqs. (2.9)-(2.12), the new maximum Hamiltonian, in terms of orbital elements, is given by:

$$\mathcal{H}^* = \mathcal{H}_0^* + \mathcal{H}_2^* + \mathcal{H}_4^*,$$

where:

$$\mathcal{H}_0^* = n\pi_M, \quad (3.24)$$
First, a brief description of Hori method is presented. For more details, see [18, 20].

Based on Lie series

In this section, a generalized canonical version of Hori method [18, 19] is applied to analyze the problem in the new formulation presented previously.

\[ \mathcal{H}_i = \frac{1}{2n^2a^2(1-e^2)} \left\{ \frac{1}{2} (1 - \cos 2\nu) \left[ 2ae\pi_\alpha + (1-e^2)\pi_\alpha \right]^2 - 2(1-e^2)\sin 2\nu \times \right. \]

\[ \left[ a\pi_\alpha \pi_\alpha + \frac{(1-e^2)}{2e} \pi_e \pi_\alpha \right] + 4(1-e^2)^{3/2} \sin \nu \left( \frac{-2e}{1+ecos\nu} + \cos \nu \right) \times \]

\[ \left[ a\pi_\alpha \pi_M + \frac{(1-e^2)}{2e} \pi_e \pi_M \right] + \frac{(1-e^2)^2}{2e^2} (1+\cos 2\nu) \pi_\alpha^2 - \frac{2(1-e^2)^{3/2}}{e^3} \]

\[ \left( \frac{-2e}{1+ecos\nu} + \cos \nu \right) \cos \nu \pi_\alpha \pi_M + \frac{(1-e^2)^3}{e^5} \left( \frac{-2e}{1+ecos\nu} + \cos \nu \right)^2 \pi_M \]

\[ + 4a^2(1-e^2)^2 \left( \frac{a}{r} \right)^2 \pi_e^2 + 4a(1-e^2)^2 \left( \frac{a}{r} \right) (\cos E + \cos \nu) \pi_e \pi_e + \]

\[ (1-e^2)^2 (\cos E + \cos \nu)^2 \pi_e^2 + 4a(1-e^2)^2 \left( \frac{a}{r} \right) \sin \nu \left( 1 + \frac{1}{1+ecos\nu} \right) \times \]

\[ \left[ \pi_\alpha \pi_\alpha - (1-e^2)^{1/2} \pi_e \pi_M \right] + \frac{2(1-e^2)^2}{e} (\cos E + \cos \nu) \left( 1 + \frac{1}{1+ecos\nu} \right) \times \]

\[ \sin \nu \left[ \pi_e \pi_\alpha - (1-e^2)^{1/2} \pi_e \pi_M \right] + \left[ \frac{(1-e^2)^3}{e} \left( 1 + \frac{1}{1+ecos\nu} \right) \times \right. \]

\[ \sin \nu \left[ \pi_\alpha - (1-e^2)^{1/2} \pi_e \pi_M \right] + \frac{1}{2} \left( \frac{r}{a} \right)^2 \cos 2(\omega + \nu) \left[ \pi_e^2 - \left( \frac{\pi \omega}{sin l} - \pi_\omega cot l \right) \right] + \]

\[ \frac{1}{2} \left( \frac{r}{a} \right)^2 \sin 2(\omega + \nu) \pi_I \left( \frac{\pi \omega}{sin l} - \pi_\omega cot l \right) \right\}, \quad (3.25) \]

\[ \mathcal{H}_j = \frac{2}{na} \frac{\partial V'_2}{\partial M} \pi_\alpha + \frac{\sqrt{1-e^2}}{nae} \left[ -\frac{\partial V'_2}{\partial \omega} + \sqrt{1-e^2} \frac{\partial V'_2}{\partial M} \right] \pi_e + \]

\[ \frac{1}{na^2 \sqrt{1-e^2} sin l} \left[ -\frac{\partial V'_2}{\partial \omega} + \cos l \frac{\partial V'_2}{\partial \omega} \right] \pi_M + \frac{1}{na^2 \sqrt{1-e^2} sin l} \frac{\partial V'_2}{\partial l} \pi_{\Omega} + \]

\[ \frac{\sqrt{1-e^2}}{na^2 e} \left[ \frac{\partial V'_2}{\partial e} - e cot l \frac{\partial V'_2}{\partial l} \right] \pi_\alpha + \frac{1}{na} \left[ -\frac{2 \partial V'_2}{\partial a} - \frac{(1-e^2)}{ae} \frac{\partial V'_2}{\partial e} \right] \pi_M, \quad (3.26) \]

where:

\[ V'_2 = \frac{\mu}{a} \left( \frac{a e}{a} \right)^2 J_2 \left\{ \left( \frac{r}{a} \right)^3 \left[ \frac{1}{2} - \frac{3}{4} \sin^2 l \right] + \frac{3}{4} \sin^2 l \left( \frac{a}{r} \right)^3 \cos 2(\omega + \nu) \right\} \quad (3.27) \]

**4 Analysis using perturbations theory**

In this section, a generalized canonical version of Hori method [18, 19] — a perturbation technique based on Lie series — is applied to analyze the problem in the new formulation presented previously. First, a brief description of Hori method is presented. For more details, see [18, 20].
4.1 Hori method

Consider a Hamiltonian function $H(x, \lambda ; \varepsilon)$ which can be developed in power series of a small parameter $\varepsilon$, in the form:

$$H(x, \lambda ; \varepsilon) = H_0(x, \lambda) + R(x, \lambda ; \varepsilon) = \lambda \cdot f(x) + \sum_{i=1}^{\infty} \varepsilon^i H_i(x, \lambda), \quad (4.1)$$

where $x, \lambda \in \mathbb{R}^n$, $H_0$ denotes the undisturbed (integrable) part of $H$ and $R$ is the disturbing part, which corresponds to the portion of $H$ that can be developed in power series.

The Hori method deals with the determination of a generating function $S(\xi, \eta; \varepsilon)$ that defines a canonical transformation $(\lambda, x) \xrightarrow{S} (\eta, \xi)$ and which also can be developed in power series of $\varepsilon$. The goal of the method is that the transformation provides a new dynamical system that is more treatable than the original one. For the problem analyzed herein, this means, for instance, to be free of periodic terms with finite frequencies (short and long period).

The transformation $S$ yields a new Hamiltonian function $\mathcal{H}(\xi, \eta; \varepsilon)$, also developed in power series of $\varepsilon$:

$$\mathcal{H}(\xi, \eta; \varepsilon) = \mathcal{H}_0(\xi, \eta) + \sum_{i=1}^{\infty} \varepsilon^i \mathcal{H}_i(\xi, \eta). \quad (4.2)$$

The new dynamical system associated to $\mathcal{H}(\xi, \eta; \varepsilon)$ is given by:

$$\frac{d\xi}{dt}^T = \frac{\partial \mathcal{H}}{\partial \eta}, \quad (4.3)$$
$$\frac{d\eta}{dt}^T = -\frac{\partial \mathcal{H}}{\partial \xi}. \quad (4.4)$$

It can be proved [18] that the general equation of the algorithm is given by:

$$\{H_0, S_m\} + \mathcal{F}_m = \mathcal{H}_m, m = 1, 2, \ldots \quad (4.5)$$

where $\{,\}$ stands for the Poisson brackets.

The problem now consists in determining the new Hamiltonian functions $\mathcal{H}_m$ and the generating functions $S_m$. In order to do that, Hori proposed an integration theory which introduces an auxiliary parameter $u$ to define what is called Hori auxiliary system:

$$\frac{d\xi}{du} = \left(\frac{\partial \mathcal{H}_0}{\partial \eta}\right)^T = f(\xi), \quad (4.6)$$
$$\frac{d\eta}{du} = -\left(\frac{\partial \mathcal{H}_0}{\partial \xi}\right)^T = -\left(\frac{\partial f}{\partial \xi}\right)^T \eta. \quad (4.7)$$

This system is integrable by hypothesis. Assuming a periodic solution, one gets, from the results of Sec. 3, that:

$$\xi = \varphi(c), \quad (4.8)$$
$$\eta = (\Delta_{\varphi}^{-1})^T b, \quad (4.9)$$

where $\Delta_{\varphi}$ is the Jacobian matrix:
\[ \Delta \phi = \left[ \frac{\partial \phi}{\partial c} \right], \] (4.10)

with \( c = [c_1 \ldots c_{n-1} \theta]^T \). The Eq. (4.5) then rewrites as:

\[ \omega \frac{dS_m}{d\theta} = \mathcal{J}_m - \mathcal{H}_m, \] (4.11)

where \( \omega \) is the frequency associated to the rapid phase \( \theta \). Finally, the application of the average principle to Eq. (4.11) yields:

\[ \mathcal{H}_m = \langle \mathcal{J}_m \rangle, \] (4.12)

\[ S_m = \frac{1}{\omega} \int \mathcal{J}_m - \langle \mathcal{J}_m \rangle \, d\theta, \] (4.13)

Similarly to the case in Sec. 3, Eqs. (4.8) and (4.9) define a Mathieu transformation between the variables \((\xi; \eta)\) and the parameters \((b; c)\),

\((\xi; \eta) \xrightarrow{\text{Mathieu}} (b; c)\).

The new dynamical system, in terms of \((b; c)\), is given by:

\[ \frac{dc^T}{dt} = \frac{\partial \mathcal{R}}{\partial b}, \] (4.14)

\[ \frac{db^T}{dt} = -\frac{\partial \mathcal{R}}{\partial c}, \] (4.15)

with \( \mathcal{R} = \mathcal{R}(c,b; \varepsilon) \).

### 4.2 Application of Hori method

Applying the Hori method described above to the Hamiltonian \( \mathcal{H}^* \) in (3.23), one gets, up to first order, the new Hamiltonian \( \mathcal{H}'^* \):

\[ \mathcal{H}'^* = \mathcal{H}'_{2}^* + \mathcal{H}'_{1}^* \] (4.16)

with:

\[ \mathcal{H}'_{2}^* = -\frac{3}{2} n J_2 \left( \frac{ae}{a} \right)^2 (1 - e^2)^{-2} \left\{ \cos I \pi \Omega + \frac{1}{2} (1 - 5 \cos^2 I) \pi \omega \right\}, \] (4.17)

\[ \mathcal{H}'_{1}^* = \frac{a}{2\mu} \left\{ 4 a^2 \pi^2 \gamma + \frac{5}{2} (1 - e^2) \pi^2 + \frac{\pi^2}{2 (1 - e^2)} \left[ (1 + \frac{3}{2} e^2) + \frac{5}{2} e^2 \cos 2 \omega \right] + \frac{5 e^2 \sin 2 \omega}{2 (1 - e^2)} \pi \Omega \left( \frac{\pi \Omega}{\sin I} - \cot I \pi \omega \right) + \frac{1}{2 (1 - e^2)} \left( \frac{\pi \Omega}{\sin I} - \cot I \pi \omega \right)^2 \times \left[ \left( 1 + \frac{3}{2} e^2 \right) - \frac{5}{2} e^2 \cos 2 \omega \right] + \frac{(5 - 4 e^2)}{2 e^2} \pi \omega \right\}. \] (4.18)

And, the generating function \( S' \) is given by:

\[ S' = S'_{2} + S'_{1}, \] (4.19)
with:

\[
S'_{J_2} = S_2 \left( \frac{a_e}{a} \right)^2 \left\{ \left( 1 - \frac{3}{2} \sin^2 I \right) \left[ \left( \frac{a_e}{r} \right)^3 - (1 - e^2)^{-3/2} \right] + \right.
\]

\[
\frac{3}{2} \sin^2 I \left( \frac{a_e}{r} \right)^3 \cos 2(\omega + \nu) \right\} \alpha \pi \Omega + \left\{ \frac{3}{4} \frac{\sin^2 I}{e(1 - e^2)} \left[ -\cos 2(\omega + \nu) - \\
e \left( \cos(2\omega + \nu) + \frac{1}{3} \cos(2\omega + 3\nu) \right) \right] + \frac{1}{e} \left[ \left( \frac{1}{2} - \frac{3}{4} \sin^2 I \right) \times \\
\left[ \left( \frac{a_e}{r} \right)^3 - (1 - e^2)^{-3/2} \right] + \frac{3}{4} \sin^2 I \left( \frac{a_e}{r} \right)^3 \cos 2(\omega + \nu) \right\} \pi e
\]

\[
+ \frac{3}{8} \frac{\sin 2I}{(1 - e^2)^2} \left[ \cos 2(\omega + \nu) + e \left( \cos(2\omega + \nu) + \frac{1}{3} \cos(2\omega + 3\nu) \right) \right] \pi I
\]

\[
\left\{ \frac{3}{2} \frac{\cos I}{(1 - e^2)^2} \left[ -(\nu - M + e \sin \nu) + \frac{1}{2} \sin 2(\omega + \nu) + \frac{e}{2} (\sin(2\omega + \nu) + \\
\frac{1}{3} \sin 2(\omega + 3\nu) \right) \right\} \pi \Omega + \left\{ \frac{3}{4} \frac{(5 \cos^2 I - 1)}{(1 - e^2)^2} (\nu - M + e \sin \nu) + \\
\frac{1}{4} \frac{(3 \cos^2 I - 1)}{e(1 - e^2)} \left[ \left( \frac{a_e}{r} \right)^2 (1 - e^2) + \left( \frac{a_e}{r} \right) + 1 \right] \sin \nu
\]

\[
+ \frac{3}{8} \frac{\sin^2 I}{e(1 - e^2)} \left[ -\left( \frac{a_e}{r} \right)^2 (1 - e^2) - \left( \frac{a_e}{r} \right) + 1 \right] \sin(2\omega + \nu)
\]

\[
\left( \left( \frac{a_e}{r} \right)^2 (1 - e^2) + \left( \frac{a_e}{r} \right) + \frac{1}{3} \right) \sin(2\omega + 3\nu) \right] + \frac{3}{8} \frac{(3 - 5 \cos^2 I)}{(1 - e^2)^2} \times \\
\left[ \sin 2(\omega + \nu) + e \left( \sin(2\omega + \nu) + \frac{1}{3} \sin(2\omega + 3\nu) \right) \right] \pi \omega \right\}. \quad (4.20)
\]
The function $\mathcal{H}^*$ in (4.16)-(4.18) represents an averaged maximum Hamiltonian, which characterizes long duration transfers, whereas $S^*$ in (4.19)-(4.21) is associated to the periodic terms of the formal solution.

5 Numerical resolution

The two-point boundary value problem associated to the transfer problem governed by the averaged Hamiltonian obtained in Sec. 4 is numerically solved using an algorithm based on the second variation theory, whose implementation is briefly discussed below.

5.1 Algorithm based on the second variation theory

The algorithm based on the second variation theory is a second order numerical method used for solving the boundary value problems (BVP’s) that arise in the resolution of optimization problems [9, 31, 32], after the application of the Pontryagin Maximum Principle. It employs the neighboring extremals concept to solve the BVP, iterating on unknown initial conditions. In its formulation, an expansion of the Hamiltonian function up to second order terms is considered, thus the name of the method. Paragraphs below briefly provide additional information on the method.

Let the following Bolza problem of optimal control: Consider the system of differential equations:

$$\frac{dx_i}{dt} = f_i(x, u), \quad i = 1, \ldots, n,$$

where $x$ is an n-vector of state variables and $u$ is an m-vector of control variables. It is assumed that there exist no constraints on the state or control variables. The problem consists in determining the optimal control $u^*(t)$ that transfers the system (5.1) from the initial conditions:
to the final conditions at $t_f$:
\[ \Psi(x(t_f)) = 0, \]  
and minimizes the performance index:
\[ J[u] = g(x(t_f)) + \int_{t_0}^{t_f} F(x,u) \, dt. \]  
The functions $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $F(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $\Psi(\cdot) : \mathbb{R}^n \to \mathbb{R}^q$, $q < n$, are assumed to be twice continuously differentiable with respect to their arguments. Furthermore, it is assumed that the matrix $[\partial \Psi / \partial x]$ has maximum rank. Initial and final times are specified.

The following two-point boundary value problem is obtained by applying the Pontryagin Maximum Principle [4] to the Bolza problem above:
\[ \begin{align*}
\frac{dx}{dt} &= H^T \lambda \\
\frac{d\lambda}{dt} &= -H^T x H u \\
H &= -F(x,u) + \lambda f(x,u),
\end{align*} \]  
with boundary conditions:
\[ \begin{align*}
x(t_0) &= x_0 \\
\lambda^T(t_f) + g_s t_f + \mu^T \Psi_s t_f &= 0 \\
\Psi(x(t_f)) &= 0,
\end{align*} \]  
where $H$ is the Hamiltonian,
\[ H = -F(x,u) + \lambda f(x,u), \]  
$\lambda \in \mathbb{R}^n$ is the adjoint vector and $\mu \in \mathbb{R}^q$ is the Lagrange multipliers vector. All quantities in (5.5)-(5.6) are evaluated on the optimal solution.

The solution of the boundary value problem defined by Eqs. (5.5)-(5.6) consists in determining the initial values of the adjoint variables $\lambda(t_0)$ and the Lagrange multipliers $\mu$. The neighboring extremals method basically consists in iteratively determining these values. The version employed in this paper uses the state transition matrix method to solve the associated linearized two-point boundary value problem obtained through the linearization of Eqs. (5.5)-(5.6) about an extremal solution [9]. The step by step computing procedure can be summarized as follows:

1. Guess the starting approximations for $\lambda(t_0)$ and $\mu$; that is, $\lambda^0(t_0)$ and $\mu^0$;

2. Integrate forward, from $t_0$ to $t_f$, the state and adjoint equations in Eq. (5.5), with the initial conditions $x(t_0) = x_0$ and $\lambda(t_0) = \lambda^0(t_0)$. The control is obtained solving $H_u = 0$;

3. Integrate forward, from $t_0$ to $t_f$, the state transition matrix differential equation:
\[ \frac{d\Phi}{dt} = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \Phi, \]  
with the initial condition $\Phi(t_0) = I$, where $A = H_{xa} - H_{ua} H_{uu}^{-1} H_{ux}$, $B = -H_{ua} H_{uu}^{-1} H_{u\lambda}$ and $C = -H_{ua} H_{uu}^{-1} H_{ux} - H_{xx}$. This step is made simultaneously with step 2;

4. Compute the matrices $T = \Psi_s \Phi_{12}(t_f)$, $V = \Psi_s^T$ and $U = \Phi_{22}(t_f) + \left( g_s^T \Psi_s + \mu^T \Psi_s \right) \Phi_{12}(t_f)$;
5. Solve the linear algebraic system:

\[ U\delta\lambda_0 + V\delta\mu = - \left( g_{xf}^T + \Psi_{xf}^T \mu + \lambda_f \right) \]

\[ T\delta\lambda_0 = -\Psi, \]

and obtain the variations \( \delta\lambda_0 \) and \( \delta\mu \);

6. Test the convergence. If it is not obtained, update the unknowns \( \lambda(\tau_0) \) and \( \mu \); that is, compute the new values \( \lambda(\tau_0) = \lambda(\tau_0) + \delta\lambda_0 \) and \( \mu = \mu + \delta\mu \);

7. Go back to step 2 and repeat the procedure until convergence is obtained.

In this paper, the method was implemented in FORTRAN, with sub-functions obtained with MAPLE 14. It was then applied to the BVP associated to the long duration transfers in Sec. 4, which is given by:

\[
\frac{da}{dt} = \frac{\partial H'}{\partial a}, \quad \frac{d\pi}{dt} = -\frac{\partial H'}{\partial a} \\
\frac{d\omega}{dt} = \frac{\partial H'}{\partial \omega}, \quad \frac{d\pi}{dt} = -\frac{\partial H'}{\partial \omega},
\]

with boundary conditions:

\[
a(t_0 = 0) = a_0 \\
\vdots \\
\omega(t_0 = 0) = \omega_0 \\
J(t_0 = 0) = 0
\]

\[
\Psi(x_f) = \begin{bmatrix} a(t_f) \\ \vdots \\ \omega(t_f) \end{bmatrix} - \begin{bmatrix} a_f \\ \vdots \\ \omega_f \end{bmatrix} = 0
\]

5.2 Results

Results were obtained for transfers between arbitrary orbits (modification of all orbital elements simultaneously) involving different changes of semimajor axis - \( a \). The remaining parameters that characterize the initial and final orbits - \( e, I, \Omega, \omega \) - were the same in all cases and chosen as: \( e_0 = 0.05, e_f = 0.25, I_0 = 10^\circ, I_f = 30^\circ, \Omega_0 = 0^\circ, \Omega_f = 30^\circ, \omega_0 = 0^\circ, \omega_f = 30^\circ \).

Computations were performed using canonical (normalized) units [29]. These units are often employed to allow, in numerical calculations, the use of simpler numbers. This is achieved by considering different standard units of distance and time. In this paper, particularly, the radius of the initial orbit is assumed as \( 1 \text{ d.u.} \) (distance unit) and the value of \( 1 \text{ t.u.} \) is such that the gravitational parameter equals \( 1 \), i.e., \( \mu = 1 \text{ d.u.}^3/\text{t.u.}^2 \).

Two values are used for the duration transfer: 100 and 200 t.u. (canonical units). Additionally, results obtained for transfers in non-central gravity field are compared to those for central Newtonian field \( (J_2 = 0) \), using the same numerical method. The objective is to verify the influence of the perturbation due to the second zonal harmonics on transfer maneuver. A weight matrix is used to control the size of corrections at each iteration. The convergence tolerance adopted is \( 1.0 \times 10^{-12} \), considering the Euclidean norm of the vectors being modified at each iteration.
Table 1  Consumption variable $J$ and relative deviations between central and non-central fields for the LP transfer problem - semimajor axis correction maneuvers

<table>
<thead>
<tr>
<th>Case</th>
<th>$t_f$(u.t.)</th>
<th>$a_0$(km)</th>
<th>$a_f/a_0$</th>
<th>$a_f$(km)</th>
<th>$e_f$</th>
<th>$J_f$(d.u.$^2$/t.u.$^3$) Central Field</th>
<th>$J_f$(d.u.$^2$/t.u.$^3$) With $J_2$</th>
<th>$\Delta J_2$(%)</th>
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</thead>
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<tr>
<td>1</td>
<td>100</td>
<td>7000</td>
<td>1.5</td>
<td>10500</td>
<td>0.25</td>
<td>1.297E-03</td>
<td>1.263E-03</td>
<td>2.7%</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>7000</td>
<td>1.5</td>
<td>10500</td>
<td>0.25</td>
<td>6.674E-04</td>
<td>6.316E-04</td>
<td>5.7%</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>7000</td>
<td>2.0</td>
<td>14000</td>
<td>0.25</td>
<td>1.399E-03</td>
<td>1.377E-03</td>
<td>1.6%</td>
</tr>
<tr>
<td>4</td>
<td>200</td>
<td>7000</td>
<td>2.0</td>
<td>14000</td>
<td>0.25</td>
<td>7.111E-04</td>
<td>6.885E-04</td>
<td>3.3%</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>10000</td>
<td>1.5</td>
<td>15000</td>
<td>0.25</td>
<td>1.280E-03</td>
<td>1.263E-03</td>
<td>1.3%</td>
</tr>
<tr>
<td>6</td>
<td>200</td>
<td>10000</td>
<td>1.5</td>
<td>15000</td>
<td>0.25</td>
<td>6.484E-04</td>
<td>6.316E-04</td>
<td>2.7%</td>
</tr>
<tr>
<td>7</td>
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<td>2.0</td>
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<td>1.388E-03</td>
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<tr>
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<td>20000</td>
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<td>6.992E-04</td>
<td>6.885E-04</td>
<td>1.5%</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>20000</td>
<td>1.5</td>
<td>30000</td>
<td>0.25</td>
<td>1.267E-03</td>
<td>1.263E-03</td>
<td>0.3%</td>
</tr>
<tr>
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<td>30000</td>
<td>0.25</td>
<td>6.356E-04</td>
<td>6.316E-04</td>
<td>0.6%</td>
</tr>
<tr>
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<td>40000</td>
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</tr>
<tr>
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<td>40000</td>
<td>0.25</td>
<td>6.911E-04</td>
<td>6.885E-04</td>
<td>0.4%</td>
</tr>
</tbody>
</table>

Fig. 1  Time history of semimajor axis for transfer case 2
Fig. 2 Time history of eccentricity axis for transfer case 2

Fig. 3 Time history of $I$, $\Omega$, and $\omega$ axis for transfer case 2
Fig. 4  Time history of semimajor axis for transfer case 5

Fig. 5  Time history of eccentricity axis for transfer case 5
Fig. 6  Time history of $I$, $\Omega$ and $\omega$ axis for transfer case 5

Fig. 7  Time history of semimajor axis for transfer case 9
Fig. 8  Time history of eccentricity axis for transfer case 9

Fig. 9  Time history of $I$, $\Omega$ and $\omega$ axis for transfer case 9
Table 1 presents values provided by the method for the consumption variable \( J \), for both central and non-central fields. Figures 1 to 9 show the time history of the main quantities for selected Cases 2, 5 and 9 of Table 1.

From Table 1 and Figs. 1 to 9, it is possible to observe several points. Firstly, greater values of \( J_f \) are associated to longer maneuvers and greater semimajor axis corrections, as expected. Additionally, there is a clear effect of \( J_2 \) on the final consumption. This effect is greater for longer transfers, comparing the equal semimajor corrections. This result is consistent, since longer maneuvers tend to propagate the effects of the simplification introduced in the assumption of central field. Moreover, the oblateness effects are lesser for maneuvers with greater \( a_0 \), since the greater the distance from the central body, the more the central field hypothesis becomes valid. Finally, there are differences in fuel consumption of up to 5%, which indicates that the oblateness effects are indeed relevant in the transfer problem considered herein.

6 Conclusions

In this paper a study of optimal low-thrust limited-power trajectories in a non-central gravity field which includes the oblateness of the Earth has been presented. The optimization problem, formulated in a Mayer form with cartesian elements, was addressed using Pontryagin’s Maximum Principle and later modified to orbital elements. A generalized canonical version of Hori method has then been applied to new formulation. The BVP associated to the averaged Hamiltonian obtained with this technique is solved, for some cases, using an algorithm based on the second variation theory. Results obtained have been compared to those for central field and showed clear effects on final consumption due to the perturbation associated to \( J_2 \). This indicates that the oblateness effects are indeed relevant.

References


