A simpler velocity quasi-potential PDE in fluid mechanics, MHD included

Richard Selescu 1

1 “Elie Carafoli” National Institute for Aerospace Research – INCAS (under the Aegis of the Romanian Academy)
Bucharest, Sector 6, Bd. Iuliu Maniu, No. 220, Code 061126, ROMANIA

* Corresponding Author Richard Selescu, E-mail address: rselescu@gmail.com

Abstract. This work studies and clarifies some local physical phenomena in fluid mechanics (and MHD), in the form of an intrinsic analytic study, regarding the PDEs of the velocity potential and (especially) 2-D “quasi-potential” (their simpler and special forms), over the virtual “isentropic” or 3-D \((V, \Omega)\) surfaces and along the “isentropic & isotachic” space curves, written for any potential and even rotational flow of an inviscid compressible fluid for both steady and unsteady motions. Using the advantages offered by the special virtual surfaces and space curves here introduced and a smart intrinsic coordinate system, a simpler PDE in only two variables, and more, a Laplace’s PDE (for any rotational pseudo-flow), was obtained, instead of the general PDE in three variables. So far, this equation was known for potential flows only. Extensions to rotational flows of a viscous compressible fluid and in MHD were given.

1 Introduction

Steichen’s vector equation (for a compressible fluid steady potential flow). Joining the continuity and physical equations to the motion equation (Euler) for an inviscid compressible fluid steady potential (isentropic) flow (of a small fluid particle), and using the local speed of sound \(a\) definition, one gets:

\[
\nabla V = \frac{V}{2} a^2 \nabla \left( V^2 \right), \quad \text{with} \quad V = \nabla \Phi, \quad \text{so obtaining (see [1])} \quad \Delta \Phi = \frac{\nabla \Phi}{2a^2} \nabla \left[ \left( \nabla \Phi \right)^2 \right]
\]

(Steichen), where, symbolically:

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\[ \Delta = \frac{1}{h_j} \sum_{j=1}^{3} \frac{\partial}{\partial x_i} \left[ \left( \sum_{j=1}^{3} h_j \right) \frac{\partial}{\partial x_i} \right] = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

and:
\[
\nabla = \sum_{i=1}^{3} \frac{k_i}{h_i} \frac{\partial}{\partial x_i} = k_x \frac{\partial}{\partial x} + k_y \frac{\partial}{\partial y} + k_z \frac{\partial}{\partial z} - \text{del (nabla)}
\]

(Laplace’s and Hamilton’s operators, respectively), in a triorthogonal system of curvilinear coordinates \(x_i; k_i\) – a 3-D basis (versors of \(Ox_i\) axes); \(h_i\) – Lamé’s coefficients; \(V\) – the local velocity of translation (of the small particle) – intensity of the local fluid field; \(V = |V|\); \(\Phi\) – the velocity potential; \(a^2 = (dp/d\rho)_{S=S_0} = \gamma R T\), with \(\gamma, R, p, \rho\) are defined later); \(S\) – the specific entropy of the fluid particle; \(T\) – the static temperature (absolute) of the fluid particle. For a rotational inviscid flow we also introduce: \(\Omega = \nabla \times \mathbf{V} = 2 \mathbf{\omega}\) – the vorticity, \(\mathbf{\omega}\) – the local velocity of rotation (of the fluid particle); \(i\) – the static specific enthalpy; \(i_0 = i + V^2/2\) – the total specific enthalpy (both of the fluid particle).

2 Steichen’s PDE of the velocity potential in an orthogonal curvilinear coordinate system

So far, to the best of the author’s knowledge, nowhere in the world literature, except for some particular cases (like the cylindrical and spherical coordinate systems), this equation was expressed. This author established Steichen’s PDE for a steady (and then unsteady) flow’s velocity potential in its general expanded form ([2] – [5]), considering a curvilinear coordinate system \(O \xi \eta \zeta\) (with \(h_{\xi}, h_{\eta}, h_{\zeta}\) – Lamé’s coefficients). The vector form of this equation (usually written for irrotational flows only) can be expanded as follows (now for rotational flows also):
\[
\nabla V_i = \frac{V_i}{a_i^2} \nabla (V_i^2) \quad \text{with} \quad V_i = \nabla \Phi_i \quad \text{but now} \quad \Omega_i \neq 0 \quad \Rightarrow \quad \Delta \Phi_i = \frac{\nabla \Phi_i \cdot \nabla (\nabla \Phi_i)}{2a_i^2}
\]

or:
\[
\Delta \Phi_i = \frac{1}{a_i^2} (\nabla \Phi_i \cdot \nabla)^2 \Phi_i \quad \text{where} \quad \Delta \Phi_i = \frac{\partial^2 \Phi_i}{\partial x^2} + \frac{\partial^2 \Phi_i}{\partial y^2} + \frac{\partial^2 \Phi_i}{\partial z^2} \quad \text{and, symbolically:}
\]
\[
(\nabla \Phi_i \cdot \nabla)^2 \Phi_i = \left[ \left( \frac{\partial \Phi_i}{\partial x} \right)^2 + \left( \frac{\partial \Phi_i}{\partial y} \right)^2 + \left( \frac{\partial \Phi_i}{\partial z} \right)^2 \right] \Phi_i = \left[ \left( \frac{\partial \Phi_i}{\partial x} \right)^2 \frac{\partial^2}{\partial x^2} + \left( \frac{\partial \Phi_i}{\partial y} \right)^2 \frac{\partial^2}{\partial y^2} + \left( \frac{\partial \Phi_i}{\partial z} \right)^2 \frac{\partial^2}{\partial z^2} \right] \Phi_i
\]

\(\Phi_i\) is the velocity quasi-potential – over the \((V, \Omega)_i\) sheets only, with \((\Omega \times V \cdot dR = T dS = 0\), with \(dR \in (V, \Omega)\) plane, due to Crocco’s equation for steady isentropic \((i_0 = \text{const.}; \nabla i = 0\) flows (see [6]): \(\Omega \times V = TVS\); \(dR\) is a virtual elementary displacement. The \((V, \Omega)_i\) sheet (envelope of the planes above, containing streamlines and vortex lines) is isentropic \((S = S_0 = \text{const.})\), allowing to introduce a quasi-potential \(\Phi_i\) (see [2] – [5] and section 10). The local speed of sound \(a_i\) is given by
\[
a_i^2 = \frac{d \rho}{d \rho} \bigg|_{S=S_0} = \gamma R T_i = \frac{\gamma - 1}{2} (W^2 - V_i^2) = \frac{\gamma - 1}{2} \left[ W^2 - \left( \frac{\partial \Phi_i}{\partial x} \right)^2 - \left( \frac{\partial \Phi_i}{\partial y} \right)^2 - \left( \frac{\partial \Phi_i}{\partial z} \right)^2 \right]
\]
where: $p$ – the fluid static pressure; $\rho$ – the fluid density; $\gamma$ – the adiabatic exponent (ratio of specific heats, $C_p / C_v$); $C_p$ is a constant (the ideal gas isobaric specific heat); $C_v$ is another constant (the ideal gas isochoric specific heat); $W$ – the gas maximum speed (corresponding to the expansion into a vacuum) – an invariant quantity (a constant).

For a perfect (an ideal) gas: $p = R \rho T$, with $R = C_p - C_v = \text{const.}$, $R$ being known as the specific ideal gas constant.

The isentropic surfaces (sheets) are analogous to D. Bernoulli’s (Lamb’s) ones ([2] – [5], [7]) for the case of a barotropic fluid ($B = V^2/2 + \int dp/\rho + gz = B_0 = \text{const.}$, with: $g$ – the acceleration of gravity; $z$ – the geometrical height (height of the considered point above a reference horizontal plane $xOy$)). All the points $M(x, y, z)$ at which this new PDE ($\Omega \neq 0$) is satisfied belong to a certain $\Omega$ isentropic sheet. In the $O\xi\eta\zeta$ curvilinear coordinate system, and in a certain orthogonal system of curvilinear coordinates $q_i (Oq_1q_2q_3)$ with the 3-D basis $k_i$, we have, resp.:

$$\Delta \Phi = \frac{1}{h_\xi h_\eta h_\zeta} \left[ \frac{\partial}{\partial \xi} \left( h_\eta h_\zeta \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( h_\zeta h_\xi \frac{\partial \Phi}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( h_\xi h_\eta \frac{\partial \Phi}{\partial \zeta} \right) \right]$$

$$\nabla \Phi = k_\xi \frac{1}{h_\xi} \frac{\partial \Phi}{\partial \xi} + k_\eta \frac{1}{h_\eta} \frac{\partial \Phi}{\partial \eta} + k_\zeta \frac{1}{h_\zeta} \frac{\partial \Phi}{\partial \zeta} = \sum_{i=1}^{3} k_i \frac{\partial \Phi}{\partial q_i}$$

$$\nabla (V^2) = \nabla \left[ \left( \nabla \Phi \right)^2 \right] = \sum_{i=1}^{3} k_i \frac{\partial}{\partial q_i} \left[ \sum_{j=1}^{3} \left( \frac{1}{h_j} \frac{\partial \Phi}{\partial q_j} \right)^2 \right]$$

$$a^2 = \frac{\gamma - 1}{2} \left( W^2 - V^2 \right) = \frac{\gamma - 1}{2} \left[ W^2 - \left( \frac{\partial \Phi}{\partial \xi} \right)^2 - \frac{1}{h_\eta^2} \left( \frac{\partial \Phi}{\partial \eta} \right)^2 - \frac{1}{h_\zeta^2} \left( \frac{\partial \Phi}{\partial \zeta} \right)^2 \right]$$

So, Steichen’s vector equation in section 1 becomes:
\[ \frac{1}{h_{\xi} h_{\eta} h_{\zeta}} \left[ \frac{\partial}{\partial \xi} \left( h_{\eta} h_{\zeta} \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( h_{\xi} h_{\zeta} \frac{\partial \Phi}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( h_{\xi} h_{\eta} \frac{\partial \Phi}{\partial \zeta} \right) \right]. \]

\[ W^2 - \frac{1}{h_{\xi}^2} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 - \frac{1}{h_{\eta}^2} \left( \frac{\partial \Phi}{\partial \eta} \right)^2 - \frac{1}{h_{\zeta}^2} \left( \frac{\partial \Phi}{\partial \zeta} \right)^2 \]

\[ = \frac{2}{\gamma - 1} \left\{ \frac{1}{h_{\xi}^2} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 + \frac{1}{h_{\eta}^2} \left( \frac{\partial \Phi}{\partial \eta} \right)^2 + \frac{1}{h_{\zeta}^2} \left( \frac{\partial \Phi}{\partial \zeta} \right)^2 \right\} \]

so, finally, the most compact symbolic expansion for the scalar equation of the velocity potential \( \Phi \) is as follows:

\[ \frac{\gamma - 1}{2} \left( \frac{1}{\prod_{i=1}^{3} h_i} \right) \sum_{i=1}^{3} \frac{\partial}{\partial q_i} \left[ \left( \frac{1}{h_i} \frac{\partial \Phi}{\partial q_i} \right) \cdot \frac{\partial \Phi}{\partial q_i} \right] \cdot \left[ W^2 - \sum_{i=1}^{3} \left( \frac{\partial \Phi}{\partial q_i} \right)^2 \right] \]

(2.1)

In order to express the right-hand side of Eqs. (1), (2) in a more compact form, we must prepare the scalar product in the right-hand side of Steichen's equation, so having:

\[ \mathbf{V} \cdot \nabla (\nabla \Phi) = \nabla \Phi \cdot \nabla [ (\nabla \Phi)^2 ] = \sum_{i=1}^{3} \frac{k_i}{h_i} \frac{\partial \Phi}{\partial q_i} \cdot \sum_{i=1}^{3} \frac{k_i}{h_i} \frac{\partial \Phi}{\partial q_i} \cdot \left[ \sum_{j=1}^{3} \left( \frac{1}{h_j} \frac{\partial \Phi}{\partial q_j} \right)^2 \right] \]

\[ = 2 \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial \Phi}{\partial q_i} \cdot \frac{\partial \Phi}{\partial q_i} \cdot \left[ \frac{\partial^2 \Phi}{\partial q_i^2} - \frac{1}{2} \left( \frac{\partial \Phi}{\partial q_i} \right)^2 \right], \]

so, finally, the compact symbolic expansion for the scalar equation of the velocity potential \( \Phi \) is as follows:

\[ \frac{\gamma - 1}{2} \left( \frac{1}{\prod_{i=1}^{3} h_i} \right) \sum_{i=1}^{3} \frac{\partial}{\partial q_i} \left[ \left( \frac{1}{h_i} \frac{\partial \Phi}{\partial q_i} \right) \cdot \frac{\partial \Phi}{\partial q_i} \right] \cdot \left[ W^2 - \sum_{i=1}^{3} \left( \frac{\partial \Phi}{\partial q_i} \right)^2 \right] \]

(2.2)

In a system of triorthogonal intrinsic coordinates \( O\zeta\eta\zeta \) tied to the isentropic \( \mathbf{V}, \mathbf{\Omega} \), surfaces (or \( O\lambda\mu\nu \)) with \( \lambda, \mu, \nu \) - lengths of the orthogonal arcs, with \( \lambda \) and \( \mu \) contained in the local plane tangent.
to \((\mathbf{V}, \Omega)\) and \(v\) directed along the normal), Laplace’s and Hamilton’s operators \((\Delta \text{ and } \nabla)\), as well as the speed of sound \(a_i\) are given by the general expressions below (\(\Phi_i\) depends on \(\xi, \eta\) and \(\zeta_{0i}\), or on \(\lambda, \mu\) and \(v_{0i}\), where: \(d\lambda = h_\xi \, d\xi\); \(d\mu = h_\eta \, d\eta\) and \(dv = h_\zeta \, d\zeta\):

\[
\Delta \Phi_i = \frac{1}{h_\xi h_\eta h_\zeta} \left[ \frac{\partial}{\partial \xi} \left( h_\eta h_\zeta \frac{\partial \Phi_i}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( h_\xi h_\zeta \frac{\partial \Phi_i}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( h_\xi h_\eta \frac{\partial \Phi_i}{\partial \zeta} \right) \right] \nonumber
\]

\[
= \left( \frac{1}{\sum_{k=1}^{3} h_k} \right) \cdot \sum_{j=1}^{3} \frac{\partial}{\partial q_j} \left[ \left( \frac{1}{h_k} \right) \cdot \frac{\partial \Phi_i}{\partial q_j} \right] ;
\]

\[
\nabla \Phi_i = k_\xi \frac{1}{h_\xi} \frac{\partial \Phi_i}{\partial \xi} + k_\eta \frac{1}{h_\eta} \frac{\partial \Phi_i}{\partial \eta} + k_\zeta \frac{1}{h_\zeta} \frac{\partial \Phi_i}{\partial \zeta} = \sum_{j=1}^{3} k_j \frac{\partial \Phi_i}{\partial q_j} ;
\]

\[
a_i^2 = \frac{\gamma - 1}{2} \left[ W^2 - \frac{1}{h_\xi^2} \left( \frac{\partial \Phi_i}{\partial \xi} \right)^2 - \frac{1}{h_\eta^2} \left( \frac{\partial \Phi_i}{\partial \eta} \right)^2 - \frac{1}{h_\zeta^2} \left( \frac{\partial \Phi_i}{\partial \zeta} \right)^2 \right] ;
\]

But \(V_{\xi i} = 0 \Rightarrow \frac{\partial \Phi_i}{\partial \zeta} = 0 \), therefore:

\[
\Delta \Phi_i = \frac{1}{h_\xi h_\eta h_\zeta} \left[ \frac{\partial}{\partial \xi} \left( h_\eta h_\zeta \frac{\partial \Phi_i}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( h_\xi h_\zeta \frac{\partial \Phi_i}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( h_\xi h_\eta \frac{\partial \Phi_i}{\partial \zeta} \right) \right] _{\zeta = \zeta_{0i}} , \text{ or:}
\]

\[
\Delta \Phi_i = \frac{1}{h_\xi} \left[ \frac{\partial}{\partial \lambda} \left( h_\eta h_\zeta \frac{\partial \Phi_i}{\partial \lambda} \right) + \frac{\partial}{\partial \mu} \left( h_\xi h_\zeta \frac{\partial \Phi_i}{\partial \mu} \right) \right] _{\nu = \nu_{0i}} ,
\]

\[
\nabla \Phi_i = k_\xi \frac{1}{h_\xi} \frac{\partial \Phi_i}{\partial \xi} + k_\eta \frac{1}{h_\eta} \frac{\partial \Phi_i}{\partial \eta} \bigg|_{\zeta = \zeta_{0i}} = k_\xi \frac{\partial \Phi_i}{\partial \lambda} + k_\eta \frac{\partial \Phi_i}{\partial \mu} \bigg|_{\nu = \nu_{0i}} ;
\]

\[
\nabla \left( V_{\xi i}^2 \right) = \nabla \left[ \left( \nabla \Phi_i \right)^2 \right] = \sum_{k=1}^{2} \frac{k_1}{h_k} \frac{\partial}{\partial q_k} \left[ \sum_{k=1}^{2} \left( \frac{1}{h_k} \frac{\partial \Phi_i}{\partial q_k} \right)^2 \right] _{q_k = q_{0k}(\zeta = \zeta_{0i})} ,
\]

\[
a_i^2 = \frac{\gamma - 1}{2} \left[ W^2 - \frac{1}{h_\xi^2} \left( \frac{\partial \Phi_i}{\partial \xi} \right)^2 - \frac{1}{h_\eta^2} \left( \frac{\partial \Phi_i}{\partial \eta} \right)^2 \right] _{\zeta = \zeta_{0i}} , \text{ or:}
\]

\[
a_i^2 = \frac{\gamma - 1}{2} \left[ W^2 - \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^2 - \left( \frac{\partial \Phi_i}{\partial \mu} \right)^2 \right] _{\nu = \nu_{0i}} .
\]

Analogously, in Steichen’s equation – a nonlinear PDE of the 2nd order in three variables – \(\xi, \eta\) and \(\zeta\) (written for a rotational flow – \(\Omega \neq 0\), but on the “isentropic surface” \(\zeta = \zeta_{0i}\)) all the terms containing the partial derivative with respect to \(\zeta\) of the quasi-potential function \(\Phi_i\) disappear \((\partial \Phi_i / \partial \zeta = 0)\) and its derivatives with respect to \(\xi, \eta\) and \(\zeta\) disappear also, thus a nonlinear PDE of the second order in only two variables – \(\xi\) and \(\eta\) – being obtained:
For the unsteady flow of a compressible fluid, the velocity potential equation has the vector form below (\( t \) – the time):

\[
\nabla \Phi - \frac{V}{2\alpha} \nabla (V^2) = \frac{1}{\alpha t} \left[ \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial}{\partial t} \left( V^2 \right) \right]
\]

with \( V = \nabla \Phi \) ; \( V, V, a \) and \( \Phi \) are now local instantaneous values;

\[
\Delta \Phi - \frac{V \Phi}{\alpha t} \nabla \left[ (\nabla \Phi)^2 \right] = \frac{1}{\alpha t} \left\{ \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial}{\partial t} \left[ (\nabla \Phi)^2 \right] \right\}
\]

or:

\[
\Delta \Phi \cdot a^2 = \frac{1}{2} V \Phi \cdot \nabla \left[ (\nabla \Phi)^2 \right] + \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial}{\partial t} \left[ (\nabla \Phi)^2 \right]
\]

This PDE also differs with respect to that for a steady flow (see section 1) by the last two terms, or in expanded form:

\[
\frac{1}{h_z h_t h_b} \left[ \frac{\partial}{\partial \xi} \left( \frac{h_t h_b}{h_z} \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{h_z h_b}{h_t} \frac{\partial \Phi}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left( \frac{h_z h_t}{h_b} \frac{\partial \Phi}{\partial \zeta} \right) \right]
\]

\[
2 C_1 (t) - \frac{1}{h_z} \left[ \frac{\partial (\Phi \xi)}{\partial \xi} \right] - \frac{1}{h_t} \left[ \frac{\partial (\Phi \eta)}{\partial \eta} \right] - \frac{1}{h_b} \left[ \frac{\partial (\Phi \zeta)}{\partial \zeta} \right] - \frac{2}{\alpha t} \left[ \frac{\partial \Phi}{\partial t} \right]
\]

\[
= \frac{2}{\gamma - 1} \left\{ \frac{1}{h_z^2} \frac{\partial (\Phi \xi)}{\partial \xi} \right\} \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{1}{h_t^2} \frac{\partial (\Phi \eta)}{\partial \eta} \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{1}{h_b^2} \frac{\partial (\Phi \zeta)}{\partial \zeta} \frac{\partial^2 \Phi}{\partial \zeta^2} + \frac{2}{\alpha t} \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi}{\partial \eta} \frac{\partial^2 \Phi}{\partial \xi \partial \eta}
\]

\[
+ \frac{1}{h_z h_t h_b} \frac{\partial \ln h_t}{\partial \xi} \frac{\partial (\Phi \xi)}{\partial \xi} + \frac{1}{h_t h_b} \frac{\partial \ln h_b}{\partial \eta} \frac{\partial (\Phi \eta)}{\partial \eta} + \frac{1}{h_z h_b} \frac{\partial \ln h_z}{\partial \zeta} \frac{\partial (\Phi \zeta)}{\partial \zeta}
\]

\[
\frac{\partial \ln h_z}{\partial \xi} \frac{\partial \Phi}{\partial \eta} \frac{\partial \Phi}{\partial \zeta} + \frac{1}{\alpha t} \left[ \frac{\partial \ln h_t}{\partial \eta} \frac{\partial \Phi}{\partial \xi} + \frac{\partial \ln h_b}{\partial \xi} \frac{\partial \Phi}{\partial \eta} + \frac{\partial \ln h_z}{\partial \zeta} \frac{\partial \Phi}{\partial \zeta} \right]
\]

\[
\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial}{\partial t} \left\{ \frac{1}{h_z^2} \left[ \frac{\partial (\Phi \xi)}{\partial \xi} \right] + \frac{1}{h_t^2} \left[ \frac{\partial (\Phi \eta)}{\partial \eta} \right] + \frac{1}{h_b^2} \left[ \frac{\partial (\Phi \zeta)}{\partial \zeta} \right] \right\}
\]

which reduces on a certain \((\mathbf{V}, \mathbf{O})\) surface \( \zeta = \zeta_0 \) to a simpler form for the 2-D velocity quasi-potential PDE:
5 Simpler forms of the 2-D velocity quasi-potential PDE over the \((V, \Omega, \lambda)\) surfaces

On a certain \(v = v_0\) isentropic surface \((V, \Omega, \lambda)\), for a steady flow, using the elementary arc lengths \(d\lambda\) and \(d\mu\), one gets:

\[
\begin{align*}
&\left[\frac{\partial^2 \Phi}{\partial \lambda^2} + \frac{\partial^2 \Phi}{\partial \mu^2} + \frac{\partial \ln(h_0 h_\xi)}{\partial \lambda} \frac{\partial \Phi}{\partial \lambda} + \frac{\partial \ln(h_0 h_\xi)}{\partial \mu} \frac{\partial \Phi}{\partial \mu}\right] \cdot \left[ W^2 - \left(\frac{\partial \Phi}{\partial \lambda}\right)^2 - \left(\frac{\partial \Phi}{\partial \mu}\right)^2 \right] \\
&= \frac{2}{\gamma - 1} \left\{ \left(\frac{\partial \Phi}{\partial \lambda}\right)^2 \frac{\partial^2 \Phi}{\partial \lambda^2} + \left(\frac{\partial \Phi}{\partial \mu}\right)^2 \frac{\partial^2 \Phi}{\partial \mu^2} + 2 \frac{\partial \Phi}{\partial \lambda} \frac{\partial \Phi}{\partial \mu} \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} - \frac{\partial \ln h_\xi}{\partial \lambda} \left(\frac{\partial \Phi}{\partial \lambda}\right)^3 \\
&\quad + \frac{\partial \ln h_\eta}{\partial \mu} \left(\frac{\partial \Phi}{\partial \mu}\right)^3 + \left(\frac{\partial \ln h_\xi}{\partial \lambda} \frac{\partial \Phi}{\partial \mu} + \frac{\partial \ln h_\eta}{\partial \lambda} \frac{\partial \Phi}{\partial \mu} \right) \left(\frac{\partial \Phi}{\partial \lambda}\right) \left(\frac{\partial \Phi}{\partial \mu}\right) \right\},
\end{align*}
\]

and the corresponding PDE of the 2-D velocity quasi-potential for the unsteady flow case becomes simpler too:

\[
\begin{align*}
&\left[\frac{\partial^2 \Phi}{\partial \lambda^2} + \frac{\partial^2 \Phi}{\partial \mu^2} + \frac{\partial \ln(h_0 h_\xi)}{\partial \lambda} \frac{\partial \Phi}{\partial \lambda} + \frac{\partial \ln(h_0 h_\xi)}{\partial \mu} \frac{\partial \Phi}{\partial \mu}\right] \cdot \left[ 2C_1(t) - \left(\frac{\partial \Phi}{\partial \lambda}\right)^2 - \left(\frac{\partial \Phi}{\partial \mu}\right)^2 - \frac{\partial \Phi}{\partial t} \right] \\
&= \frac{2}{\gamma - 1} \left\{ \left(\frac{\partial \Phi}{\partial \lambda}\right)^2 \frac{\partial^2 \Phi}{\partial \lambda^2} + \left(\frac{\partial \Phi}{\partial \mu}\right)^2 \frac{\partial^2 \Phi}{\partial \mu^2} + 2 \frac{\partial \Phi}{\partial \lambda} \frac{\partial \Phi}{\partial \mu} \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} - \frac{\partial \ln h_\xi}{\partial \lambda} \left(\frac{\partial \Phi}{\partial \lambda}\right)^3 \\
&\quad + \frac{\partial \ln h_\eta}{\partial \mu} \left(\frac{\partial \Phi}{\partial \mu}\right)^3 + \left(\frac{\partial \ln h_\xi}{\partial \lambda} \frac{\partial \Phi}{\partial \mu} + \frac{\partial \ln h_\eta}{\partial \lambda} \frac{\partial \Phi}{\partial \mu} \right) \left(\frac{\partial \Phi}{\partial \lambda}\right) \left(\frac{\partial \Phi}{\partial \mu}\right) \right\} \\
&+ \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial}{\partial t} \left(\frac{\partial \Phi}{\partial \xi}\right)^2 + \left(\frac{\partial \Phi}{\partial \mu}\right)^2.
\end{align*}
\]

Choosing the \(\xi\) intrinsic coordinate along the streamlines direction (assumed as being known), the new PDE becomes an ODE \((\partial \Phi_\xi/\partial \xi = d \Phi_\xi/d \xi)\) of the 2nd order:

\[
\frac{1}{h_\xi^2} \left[ \frac{d^2 \Phi}{d \xi^2} + \frac{\partial \ln(h_0 h_\xi)}{\partial \xi} \frac{d \Phi}{d \xi} \right] \cdot \left[ W^2 - \frac{1}{h_\xi^2} \left(\frac{d \Phi}{d \xi}\right)^2 \right] = \frac{2}{\gamma - 1} \left(\frac{d \Phi}{d \xi}\right)^2 \frac{d^2 \Phi}{d \xi^2} - \frac{\partial \ln h_\xi}{\partial \xi} \left(\frac{d \Phi}{d \xi}\right)^3,
\]

representing the simplest form for the equation of the now 1-D velocity quasi-potential \(\Phi_\xi\) for a steady flow, and more, using the elementary arc length \(d \lambda = h_\xi d \xi\):
thus the streamlines being just the characteristic lines. Analogously, for an unsteady flow, the new PDE below becomes the simplest one also (Φ depends on ξ and t):

\[
\begin{aligned}
\frac{\gamma - 1}{2h^2_\xi} \left( \frac{\partial^2 \Phi_i}{\partial \xi^2} + \frac{\partial \ln(h_\eta h_\zeta)}{\partial \lambda} \frac{\partial \Phi_i}{\partial \lambda} \right) \cdot \left[ W^2 - \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^2 \right] = & \frac{2}{\gamma - 1} \left[ \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^2 + \frac{\partial \ln h_\xi}{\partial \lambda} \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^3 \right] = \\
\end{aligned}
\]

and more, with the elementary arc length dλ = h_ξ dξ:

\[
\begin{aligned}
\frac{\gamma - 1}{2} \left( \frac{\partial^2 \Phi_i}{\partial \lambda^2} + \frac{\partial \ln(h_\eta h_\zeta)}{\partial \lambda} \frac{\partial \Phi_i}{\partial \lambda} \right) \cdot \left[ 2C_i(t) - \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^2 \right]
= & \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^2 + \frac{\partial \ln h_\xi}{\partial \lambda} \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^3 + \frac{\partial^2 \Phi_i}{\partial t^2} + \frac{\partial}{\partial t} \left[ \frac{1}{h^2_\xi} \left( \frac{\partial \Phi_i}{\partial \lambda} \right)^2 \right].
\end{aligned}
\]

6 Special forms of the 2-D velocity quasi-potential PDE along the quasi-Laplace (elliptic) space lines of a compressible fluid rotational flow

On a certain virtual “i” surface, for an inviscid fluid flow we have:

\[
\nabla (V^2/2)|_i + (\Omega \times \mathbf{V})|_i = -(\nabla p/\rho)|_i.
\]

Performing a scalar multiplication of this relation by a virtual elementary displacement d\mathbf{R}_i \in (\mathbf{V}, \Omega)|_i (therefore an isentropic virtual surface), we obtain:

\[
d(V^2/2)|_i = -(dp/\rho)|_i, \text{ with } p = K_d \rho^\gamma,
\]

so a first integrable form. On the other hand, over any polytropic virtual surface we have:

\[
(p/\rho^n)|_j = (p/\rho_n)|_i = \text{const.}, \text{ or } p = \text{const.}_j \rho_n, \text{ and so }:
\]

\[
\nabla p = \text{const.}_j \cdot \nabla (\rho^n) = \text{const.}_j \cdot \rho^{n-1} \nabla \rho.
\]

Performing a scalar multiplication of this relation by a virtual elementary displacement d\mathbf{R}_j contained in the plane tangent to the polytropic “j” surface, we obtain:

\[
dp_j = \text{const.}_j \cdot (\rho^{n-1} dp)|_j.
\]

So, along the intersection “ij” lines (with: d\mathbf{R}_{ij} = k\nabla S|_i \times \nabla (p/\rho^n)|_j) of the two surface families we have:

\[
d(V^2/2)|_{ij} = -(dp/\rho)|_{ij} = -\text{const.}_j (\rho^{n-2}dp)|_{ij}.
\]

Analogously we have:
Taking into account that through any intersection “ij” line of the two surface families above there is a “star (pencil) of sheets” passing (for various particular values of the polytropic exponent “n”, e.g.: isentropic, isothermal & isotachic, isobaric, isochoric and a general polytropic one), we can write along these lines:
\[ dS_{ij} = d(V^2/2)_{ij} = -(dp/p)_{ij} = -\text{const.}(p^{n-2}dp)_{ij} = -\text{const.}[(T^{1/(n-1)}dT)/p]_{ij} = 0, \]
or simpler:
\[ dS_{ij} = dT_{ij} = dV_{ij} = dp_{ij} = d\rho_{ij} = d(p/p^n)_{ij} = 0, \] so having:
\[ dR_{ij} = k \cdot \nabla S \times \nabla T = k_1 \cdot \nabla S \cdot \nabla V = k_2 \cdot \nabla S \times \nabla p = k_3 \cdot \nabla S \times \nabla \rho = k_4 \cdot \nabla S \times \nabla (p/p^n). \]
Through any point of the considered rotational flow, such of “star of integral sheets” is passing.

Applying a scalar multiplication of Steichen’sPDE of the compressible 2-D velocity “quasi-potential” \( \Phi_{ij} \),
\[ \nabla V_{ij} = \dot{V}_{ij} \cdot \nabla (V^2_i)/2a_i^2 \quad \text{(with \( V_{ij} = \nabla \Phi_{ij} \)) by the \( dR_{ij} \) above, one gets}
\]
\[ \nabla V_{ij} \cdot dR_{ij} = \dot{V}_{ij} \cdot d(V^2_i)/2a_i^2 \quad \text{with \( d(V^2_i) = 0 \), and so \( \nabla V_{ij} = 0 \) (because \( dR_{ij} \neq 0 \)).}
\]

Introducing the scalar function \( \Phi_{ij}(\xi, \eta) \) one obtains a PDE identical to Laplace’s one: \( \Delta \Phi_{ij} = 0 \), so \( \Phi_{ij}(\xi, \eta) = 0 \), being now a harmonic function. This simpler PDE is valid for a certain rotational flow \( (V_{ij} = \nabla \Phi_{ij}; \quad \omega_{ij} = \nabla \times V_{ij} \neq 0; \quad (p/p^n)_{ij} = \text{const.}) \) - a “quasi-incompressible” fluid behavior along the intersection lines of any isentropic virtual surface with any polytropic one. Over flow’s isothermal & isotachic virtual surfaces we have: \( |V| = V = \text{const.} \) or \( \nabla V \cdot dR = dV = 0 \) (the fluid has a “quasi-uniform” behavior).

The vector equation \( V \cdot \nabla (V^2) = 0 \) (a zero-scalar product in the right-hand side of Steichen’s equation), expressed, say, in the Cartesian system in the expanded form below:
\[ V_x^2 \frac{\partial V_x}{\partial x} + V_y^2 \frac{\partial V_y}{\partial y} + V_z^2 \frac{\partial V_z}{\partial z} + V_x V_y \left( \frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) +
V_y V_z \left( \frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y} \right) + V_z V_x \left( \frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z} \right) = 0 , \]
(not implying the existence of a velocity potential \( \Phi \)) has the solutions:
1) \( V = 0 \) \( (V_x = V_y = V_z = 0) \), representing a trivial solution:
   a) the equilibrium (fluid statics) case, and
   b) the flow stagnation lines \( (p = p_0; \quad T = T_0) \), for the plane flows and for some special axisymmeric flows;
2) \( \nabla (V^2) = 0 \), where \( \nabla (V^2) = 2V \cdot \nabla V = 0 \) the acceleration being now \( \mathbf{a} = \omega \times \mathbf{V} = 2\omega \times \mathbf{V} = TVS \), like a Coriolis one the vectors \( \mathbf{V} \), \( \omega \) and \( \mathbf{a} \) forming thus an ortogonal system.
   a) the flow of an incompressible fluid \( (p = \text{const.} \Rightarrow \nabla p = 0, \) and so: \( \nabla (V^2) = -2a^2 \cdot \nabla p/p = 0 \)) - a liquid flow model;
   b) a parallel and uniform flow of a compressible fluid \( (V = \text{const.}, \) with \( \omega = 0 \)) - a trivial solution also, and
   c) the flow’s isotachic surfaces \( \nabla V = 0 \), and especially \( d(V^2_i) = 0 \) (along the intersection “ij” lines above), which seems to be the most important solution (Selescu), due to the fact that over any “i” isentropic surface we have \( V_i = \nabla \Phi_i \), and so \( V_{ij} = \nabla \Phi_{ij} \) also (a quasi-potential flow);
3. \( \mathbf{V} \perp \nabla \mathbf{V} \), or \( \mathbf{V} \perp \nabla (\mathbf{V}^2) \) - a very particular solution; \( \nabla (\mathbf{V}^2) \perp \) (the isotachic & isothermal surfaces), hence the velocity \( \mathbf{V} \) lies just on (is tangent to) the isotachic & isothermal surfaces (having no component along the normal), representing flows due to vortex distributions:

a) the plane flow of a compressible fluid due to a straight infinite vortex filament, having two regions: subsonic and supersonic, separated by a circular critical line, along which the Mach number reaches the value 1, all streamlines being concentric circles, including a solid cylindrical nucleus (a no-motion region); on its surface the maximum speed \( W \) is reached (see fig. 1);

b) the superposition of the flow above with a uniform flow, parallel to vortex filament, all streamlines being co-axial circular cylindrical helices with the same pitch;

c) the same as above, but with a certain direction of the uniform parallel flow;

d) some flows of an inviscid compressible fluid due to an infinite sequence of very close vortex filaments, leading to streamlines patterns identical (and velocity distributions somehow related) to those for the well-known models of plane and axisymmetric flows of a viscous fluid, like plane and circular Couette, and Hagen–Poiseuille ones; for an extension to the viscous flow of a Newtonian fluid see sections 10 and 11). The vector field given in figure 1 corresponds to the velocity field due to a straight infinite vortex filament directed along the z-axis (normal to the x, y-plane):

\[
\mathbf{V}(x, y, z) = \frac{\Gamma}{2\pi} \left(-k_x \frac{y}{x^2+y^2} + k_y \frac{x}{x^2+y^2} + k_z \cdot 0\right); \quad |\mathbf{V}(x, y, z)| = V = \frac{\Gamma}{2\pi} \cdot \frac{1}{(x^2+y^2)^{1/2}} = \frac{\Gamma}{2\pi} \cdot \frac{1}{R},
\]

\( \Gamma \) is the vortex constant intensity (the velocity circulation). The lengths of the velocity vectors induced by the vortex filament are proportional to their intensities (moduli) – a hyperbolic variation law.

Their directions are tangent to the concentric circles having the center in the origin, and their senses are represented by arrowheads, corresponding to a left-hand screw (counter-clockwise). Along a certain circle (streamline) \( R = (x^2+y^2)^{1/2} = \text{const.} \), the velocity vector has a constant modulus. So, these circles are the flow isotachs (isotachic lines). The critical speed \( c \) (for which \( V = a \)) is reached along a circle of radius \( R_c = \Gamma/2\pi c = \Gamma/2\pi \cdot \left((\gamma+1)\rho_0/2\gamma\rho_0\right)^{1/2} \), and the maximum speed
W (for which \( V = W \)) is reached along another circle (the wall of the solid cylindrical nucleus – the no-motion region), of minimum radius \( R_W = \Gamma / 2 \pi W = \Gamma / 2 \pi \cdot (\gamma - 1) \rho_0 / 2 \gamma \rho_0)^{1/2} \), so that for the region \( R \in (R_W, R_c) \) the plane flow is supersonic (\( V \) decreases from \( W \) to \( c \)), and for the semi-infinite region \( R > R_c \) the flow is subsonic (\( V \) decreases from \( c \) to 0). The above field includes a vortex at its center (singularity), so it is rotational. However, any simply-connected subset that excludes the vortex line will have zero curl, \( \mathbf{\Omega} = 0 \) (no vorticity), the fluid particles performing circular translations only. We give below a detailed explanation of solutions 2 and 3. Intersecting the isentropic \( (\nabla \cdot \mathbf{dR} = 0) \) virtual surfaces \( S = S_0 = \text{const.} \cdot \lambda \) with the special isotachic ones \( (|V| = V = V_j = \text{const.},\ j) \) one obtains a family (net) of space curves along which Steichen’s PDE of the compressible 2-D velocity “quasi-potential” \( \Phi(\xi, \eta) \) becomes again simpler, identical to Laplace’s one: \( \Delta \Phi_{ij} = 0 \), \( \Phi_{ij}(\xi, \eta) \) being now a harmonic function.

So, the solution \( \mathbf{V}_{ij} \) of the general Steichen’s PDE \( \nabla \mathbf{V}_{ij} = \mathbf{V}_{ij} \cdot \nabla (V_j^2) / 2 i j^2 \), written for a special rotational flow \( \mathbf{V}_{ij} = \nabla \Phi_{ij}; \mathbf{\Omega}_{ij} = \nabla \times \mathbf{V}_{ij} \neq 0; |\mathbf{V}_{ij}| = V_{ij} = \text{const.}, \ j \), with \( \mathbf{V}_{ij} \cdot \nabla (V_j^2) = 0 \), is now the solution of the system:

\[
\begin{align*}
\nabla \mathbf{V}_{ij} & = \mathbf{V}_{ij} \cdot \nabla (V_j^2) / 2 a_i^2, \quad \text{with} \quad \mathbf{V}_{ij} = \nabla \Phi_{ij}(\xi, \eta); \\
\nabla \times \mathbf{V}_{ij} & = 0,
\end{align*}
\]

or, along the “\( ij \)” lines \( \mathbf{V}_{ij} = \mathbf{V}_{ij} \), but not directed along the “\( ij \)” lines; \( a_i = a_j = a_{ij} \); \( \Phi_{ij} = \Phi_{ij} \):

\[
\begin{align*}
\nabla \mathbf{V}_{ij} & = \mathbf{V}_{ij} \cdot \nabla (V_j^2) / 2 a_i^2, \quad \text{with} \quad \mathbf{V}_{ij} = \nabla \Phi_{ij}(\xi, \eta); \\
\nabla \times \mathbf{V}_{ij} & = 0,
\end{align*}
\]

or, mainly:

\[
\begin{align*}
\Delta \Phi_{ij} & = 0; \\
\text{d}(V_j^2) & = 0, \quad (6.1)
\end{align*}
\]

this leading to \( \nabla \mathbf{V}_{ij} = 0 \) and so obtaining the simpler form: \( \nabla \mathbf{V}_{ij} = \Delta \Phi_{ij} = 0 \); with \( \mathbf{V}_{ij} = \nabla \Phi_{ij} = (\nabla \times \mathbf{\Psi}_{ij}) / \rho \), with \( \mathbf{\Psi}_{ij} \neq \nabla G_{ij} \) - the 3-D stream function vector (Selescu, see section 7 in [7], for the mass flux density vector \( \rho \mathbf{V}_{ij} \)):

\[
\begin{align*}
\Delta \Phi_{ij} & = \frac{1}{h^2 h \eta h^2} \left[ \frac{\partial}{\partial \xi} \left( \frac{h \eta h \xi}{h \xi} \frac{\partial \Phi_{ij}}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{h \eta h \xi}{h \eta} \frac{\partial \Phi_{ij}}{\partial \eta} \right) \right]; \quad \left\{ \begin{array}{c}
\zeta = \zeta_0; \\
\mathbf{V} = \mathbf{V}_j
\end{array} \right. 0 , \quad \text{or}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial \xi} \left( \frac{h \eta h \xi}{h \xi} \frac{\partial \Phi_{ij}}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \frac{h \eta h \xi}{h \eta} \frac{\partial \Phi_{ij}}{\partial \eta} \right); \quad \left\{ \begin{array}{c}
\zeta = \zeta_0; \\
\mathbf{V} = \mathbf{V}_j
\end{array} \right. 0 ,
\end{align*}
\]

\[
\begin{align*}
\frac{1}{h^2 \xi} \frac{\partial^3 \Phi_{ij}}{\partial \xi^2} \xi + \frac{1}{h^2 \eta} \frac{\partial^3 \Phi_{ij}}{\partial \eta^2} \eta + \frac{1}{h \eta} \frac{\partial \ln(h \eta h \xi)}{\partial \xi} \frac{\partial \Phi_{ij}}{\partial \xi} + \frac{1}{h \eta} \frac{\partial \ln(h \eta h \xi)}{\partial \eta} \frac{\partial \Phi_{ij}}{\partial \eta}; \quad \left\{ \begin{array}{c}
\zeta = \zeta_0; \\
\mathbf{V} = \mathbf{V}_j
\end{array} \right. 0 ;
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 \Phi_{ij}}{\partial \xi^2} + \frac{\partial^2 \Phi_{ij}}{\partial \eta^2} + \frac{\partial \ln(h \eta h \xi)}{\partial \xi} \frac{\partial \Phi_{ij}}{\partial \xi} + \frac{\partial \ln(h \eta h \xi)}{\partial \eta} \frac{\partial \Phi_{ij}}{\partial \eta}; \quad \left\{ \begin{array}{c}
\zeta = v_0; \\
\mathbf{V} = \mathbf{V}_j
\end{array} \right. 0 ,
\end{align*}
\]
for a steady flow, therefore an elliptic PDE, irrespective of the pseudo-flow character (subsonic or supersonic), this representing the annulment of the first factor (inside the first square brackets) in the left-hand side of PDE (3). For an unsteady flow Steichen’s equation takes the form

$$\nabla \mathbf{V}_{ij} - \frac{V_{ij}}{2a_{ij}} \nabla (V_{ij}^2) = \frac{1}{\alpha_{ij}} \left[ \frac{\partial^2 \Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t} \left( V_{ij} \right) \right],$$

with: \( V_{ij} \cdot \nabla (V_{ij}^2) = 0 \), this resulting in:

$$\Delta \Phi_{ij} = \frac{1}{\alpha_{ij}} \left\{ \frac{\partial^2 \Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t} \left[ (\nabla \Phi_{ij})^2 \right] \right\},$$

or:

$$\Delta \Phi_{ij} \cdot a_{ij} = \frac{\partial^2 \Phi_{ij}}{\partial t^2} + \frac{\partial}{\partial t} \left[ (\nabla \Phi_{ij})^2 \right],$$

with:

$$a_{ij}^2 = \frac{\gamma - 1}{2} \left[ 2C_i(t) - \frac{1}{h_{\xi}} \left( \frac{\partial \Phi_{ij}}{\partial \xi} \right)^2 - \frac{1}{h_{\eta}} \left( \frac{\partial \Phi_{ij}}{\partial \eta} \right)^2 - 2 \frac{\partial \Phi_{ij}}{\partial t} \right],$$

so obtaining along the space curves \((S = S_{\text{inc}}; V = V_{ij})\), i.e. coupling the cases 4 and 5 in subsection 1.2 in [7]), for the function \( \Phi_{ij} (\xi, \eta, t) \) the following PDE:

$$\left\{ \begin{array}{l}
\frac{1}{h_{\xi}} \frac{\partial^2 \Phi_{ij}}{\partial \xi^2} + \frac{1}{h_{\eta}} \frac{\partial^2 \Phi_{ij}}{\partial \eta^2} + \frac{1}{h_{\xi}} \frac{\partial}{\partial \xi} \ln(h_{\xi} h_{\zeta}) \frac{\partial \Phi_{ij}}{\partial \xi} + \frac{1}{h_{\eta}} \frac{\partial}{\partial \eta} \ln(h_{\eta} h_{\zeta}) \frac{\partial \Phi_{ij}}{\partial \eta} + \frac{1}{h_{\xi}} \frac{\partial}{\partial \xi} \ln(h_{\zeta} h_{\eta}) \frac{\partial \Phi_{ij}}{\partial \xi} + \frac{1}{h_{\eta}} \frac{\partial}{\partial \eta} \ln(h_{\xi} h_{\eta}) \frac{\partial \Phi_{ij}}{\partial \eta} = 0, \\
2C_i(t) - \frac{1}{h_{\xi}} \left( \frac{\partial \Phi_{ij}}{\partial \xi} \right)^2 - \frac{1}{h_{\eta}} \left( \frac{\partial \Phi_{ij}}{\partial \eta} \right)^2 - 2 \frac{\partial \Phi_{ij}}{\partial t} = 0,
\end{array} \right.$$
which must be also considered for solving the system (6.1), besides the boundary conditions of the respective rotational flow, along an “ij” space line, lying on an isentropic “i” surface and an isothermal & isotachic “ii” one.

7 Special forms of the 2-D velocity quasi-potential PDE along other (hyperbolic & parabolic) space lines of a compressible fluid rotational flow

Intuitively, we can put the existence problem for a family of space curves passing through any point also, along which the velocity quasi-potential function $\Phi_{ij}$ respects another rule, e.g.: the wave (vibrating string) PDE of the 2nd order (in one space dimension), instead of Laplace’s one, that means being of a hyperbolic type, just in its canonical form:

$$\partial^2 \Phi_{ij}(\lambda, \mu, v_{ii})/\partial \lambda \partial \mu + \ldots \ (\text{lower order terms}) = 0,$$

irrespective of the pseudo-flow character (subsonic or supersonic), in the same $\Omega \lambda \mu V$ smart intrinsic triorthogonal curvilinear coordinate system previously used. But we will treat this problem, for simplicity reasons, in the classical Cartesian system, at least for the beginning, to understand the new proposed mechanism for solving the above problem, this meaning to treat a true potential flow (throughout). In the case of a rotational flow, like previously, the searched for space curves must be obtained as being the intersection lines of two surface families:

1. the isentropic ($V_i, \Omega_i$) ones, allowing us to introduce a 2-D velocity quasi-potential $\Phi_i$; and
2. the new special surfaces (like previous isotachic & isothermal ones) over which another condition must be satisfied. Let us write Steichen’s equation in the form (also see section 1):

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} - \frac{1}{a^2} \left[ V_x^2 \frac{\partial V_x}{\partial x} + V_y^2 \frac{\partial V_y}{\partial y} + V_z^2 \frac{\partial V_z}{\partial z} \right] = 0,$$

with:

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_z}{\partial z} = V; \quad a^2 = \frac{\gamma - 1}{2} \left[ W^2 - (V_x^2 + V_y^2 + V_z^2) \right] = \frac{\gamma - 1}{2} (W^2 - V^2); \quad \text{and}

V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_z}{\partial z} = \frac{1}{3} \left( \frac{\partial V_x^3}{\partial x} + \frac{\partial V_y^3}{\partial y} + \frac{\partial V_z^3}{\partial z} \right) = \frac{1}{3} \nabla V_3,$$

thus obtaining a new special form for this equation:

$$\nabla V - \frac{2}{\gamma - 1} \frac{1}{W^2 - V^2} \left( \frac{1}{3} \nabla V_3 + V_x \frac{\partial V_x}{\partial y} + V_y \frac{\partial V_y}{\partial x} \right) + V_y \frac{\partial V_y}{\partial z} \right) + V_z \frac{\partial V_z}{\partial y} = 0. \quad (7.1)$$

Now we impose (consider the possibility to satisfy) the condition:

$$\nabla V - \nabla V_3/3a^2 = 0, \quad or \ a^2 \nabla V = \nabla V_3/3, \quad (7.2)$$

or in a more suggestive form: $\nabla V_3/V(3V) = a^2(V)$ – the ratio of two divergences (in this case), condition giving the new searched for special surfaces, concurrently further satisfying Steichen’s PDE (7.1) above:

$$a^2 \nabla V = \nabla V(V^2)/2, \quad \text{or, equating with that given by Eq. (7.2):}$$
3∇(V^2) = 2∇V_3(\neq 0), \quad (7.3)

(not depending on \(a\)). Let us write Eq. (7.2) in a scalar form:

\[ V_x V_y \left( \frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) + V_y V_z \left( \frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y} \right) + V_x V_z \left( \frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x} \right) = 0, \quad (7.4) \]

generally valid along the intersection lines of the isentropic “i” surfaces with the “j” special ones satisfying Eq. (7.2). These lines are solutions of the system formed by Eqs. (7.1) and (7.2). If \(\nabla V_3 = 0\), the problem reduces to that in section 6. Introducing the velocity potential \(\Phi\),

\[ V = \nabla \Phi, \quad \text{and:} \quad V_x = \partial \Phi/\partial x; \quad V_y = \partial \Phi/\partial y; \quad V_z = \partial \Phi/\partial z; \quad \nabla \Phi = \Delta \Phi, \]

we obtain for the condition Eq. (7.2) the expanded equivalent forms (the PDE of the new searched for “j” surfaces):

1. \((a^2 - V_x^2) \frac{\partial V_x}{\partial x} + (a^2 - V_y^2) \frac{\partial V_y}{\partial y} + (a^2 - V_z^2) \frac{\partial V_z}{\partial z} = 0\), also expressing the orthogonality of the vectors \(V_2\) and \(\delta V\), defined as follows: \(V_2 = k_x (a^2 - V_x^2) + k_y (a^2 - V_y^2) + k_z (a^2 - V_z^2)\); \(\delta V = k_x \frac{\partial V_x}{\partial x} + k_y \frac{\partial V_y}{\partial y} + k_z \frac{\partial V_z}{\partial z}\), \((V_2 \cdot \delta V = 0)\);

2. \(a^2 \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) = \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \),

or more (with \(a^2\) previously defined as a function of \(V_x, V_y\) and \(V_z\)):

\[ 2' \begin{bmatrix} W^2 - \frac{\partial \Phi}{\partial x} \left( \frac{\partial \Phi}{\partial x} \right)^2 - \left( \frac{\partial \Phi}{\partial y} \right)^2 - \left( \frac{\partial \Phi}{\partial z} \right)^2 \frac{\partial^2 \Phi}{\partial x^2} \\ W^2 - \frac{\partial \Phi}{\partial y} \left( \frac{\partial \Phi}{\partial y} \right)^2 - \left( \frac{\partial \Phi}{\partial x} \right)^2 - \left( \frac{\partial \Phi}{\partial z} \right)^2 \frac{\partial^2 \Phi}{\partial y^2} \\ W^2 - \frac{\partial \Phi}{\partial z} \left( \frac{\partial \Phi}{\partial z} \right)^2 - \left( \frac{\partial \Phi}{\partial x} \right)^2 - \left( \frac{\partial \Phi}{\partial y} \right)^2 \frac{\partial^2 \Phi}{\partial z^2} \end{bmatrix} = 0 \]

and for previous Eq. (7.4) (using Schwarz’ theorem):

\[ 2 \left( \frac{\partial \Phi}{\partial x} \frac{\partial^2 \Phi}{\partial y \partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial z \partial y} + \frac{\partial \Phi}{\partial z} \frac{\partial^2 \Phi}{\partial x \partial z} \right) = 0, \]

just the sum of until now negligible terms with respect to \(a^2 \Delta \Phi\) in the velocity potential PDE for small perturbations. This result is very important and leads for the case of a 2-D plane motion (\(V_z = \partial \Phi/\partial z = 0\)) to the simplest PDE of the velocity potential \(\Phi\) (a very particular situation):

\[ \frac{\partial^2 \Phi}{\partial x \partial y} = 0, \quad (7.5) \]

(meaning: \(\partial V_x/\partial y = \partial V_y/\partial x = 0\), leading to a special plane flow: \(V_x = f_1(x); \quad V_y = f_2(y)\) and \(\Phi = \int f_1(x) dx + \int f_2(y) dy = \Phi_1(x) + \Phi_2(y)\), like that in a channel with hyperbolic walls, or a right-angled corner), representing the canonical form of a 2nd order PDE of a hyperbolic type with constant coefficients. The condition Eq. (6) becomes now:
A Simpler Velocity Quasi-Potential PDE in Fluid Mechanics, MHD included

8 A simple example: the 3-D conical supersonic flow (rotational, i.e. non-isentropic)

An interesting example of application was given in [8], establishing the general ODE (1) of the velocity quasi-potential \( \Phi_t = RV(\phi, \chi_0) = RV_0(\phi) \) – on every “i” isentropic sheet \( \chi = \chi_0 \) for any (rotational, therefore non-isentropic) 3-D conical flow, writing and analyzing it in the new smart intrinsic generalized spherical (conical) triorthogonal coordinate system \((R, \phi, \chi)\):

\[
(\gamma - 1) \{ V_i'' + \left[ \ln |f(\phi)| \right]' \cdot V_i' + 2V_i \} \left( W^2 - V_i^2 - V_j^2 \right) - 2 (V_i'' + V_i) V_i'^2 = 0 ,
\]

with \( ' = \frac{d}{d\phi} \) and \( '' = \frac{d^2}{d\phi^2} \),

instead of the usually written (approximate) PDE of the velocity potential \( \Phi = RV_\phi(\theta, \omega) \), with: \( V = V_p + V_r \) (Helmholtz) where: \( V_p = \nabla \Phi \) and \( V_r = \nabla \times \Psi \) (a solenoidal field), or (see [9]) \( V_r = \Phi_1 \nabla \Phi_2 \) (a biscalare field)
\[
\left(1 - \frac{V_{\theta p}^2}{a^2}\right) \frac{\partial V_{\theta p}}{\partial \theta} + \left(1 - \frac{V_{\varphi p}^2}{a^2}\right) \frac{1}{\sin \theta} \frac{\partial V_{\varphi p}}{\partial \omega} + V_{kp} \left(2 + \frac{V_{\theta p}^2 + V_{\varphi p}^2}{a^2}\right) - \frac{2V_{\theta p}V_{\varphi p}}{a^2\sin \theta} \frac{\partial V_{\theta p}}{\partial \omega} + \cot \theta \left(1 + \frac{V_{\varphi p}^2}{a^2}\right) V_{\theta p} = 0 ,
\]

or with \( \Phi = R \cdot V_p(\theta, \omega) \) : \( V_{kp} = \frac{\partial \Phi}{\partial R} = V_p ; \quad V_{\theta p} = \frac{1}{R} \frac{\partial \Phi}{\partial \theta} = \frac{\partial V_p}{\partial \theta} ; \quad V_{\varphi p} = \frac{1}{R\sin \theta} \frac{\partial \Phi}{\partial \omega} = \frac{1}{\sin \theta} \frac{\partial V_p}{\partial \omega} \), so obtaining:

\[
\left[ 1 - \frac{1}{a^2} \left( \frac{\partial V_p}{\partial \theta} \right)^2 \right] \frac{\partial^2 V_p}{\partial \theta^2} + \left[ 1 - \frac{1}{a^2} \sin^2 \theta \left( \frac{\partial V_p}{\partial \omega} \right)^2 \right] \frac{1}{\sin^2 \theta} \frac{\partial^2 V_p}{\partial \omega^2} + \left\{ 2 + \frac{1}{a^2} \left[ \left( \frac{\partial V_p}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial V_p}{\partial \omega} \right)^2 \right] \right\} V_p - \frac{2}{a^2} \frac{\partial V_p}{\partial \varphi} \frac{\partial V_p}{\partial \omega} \frac{\partial^2 V_p}{\partial \theta \partial \omega} + \cot \theta \left[ 1 + \frac{1}{a^2} \sin^2 \theta \left( \frac{\partial V_p}{\partial \omega} \right)^2 \right] \frac{\partial V_p}{\partial \theta} = 0 \quad \text{(see, e.g.: [10])} ,
\]

with:

\[
a^2 = \frac{\gamma - 1}{2} \left( W^2 - V_p^2 - V_{\varphi p}^2 \right) \cong \frac{\gamma - 1}{2} \left( W^2 - \hat{V}_{kp} - V_{\varphi p} - V_{\varphi p} \right) = \frac{\gamma - 1}{2} \left( W^2 - V_p^2 - V_{\varphi p}^2 - \left( \frac{\partial V_p}{\partial \theta} \right)^2 - \frac{1}{\sin^2 \theta} \left( \frac{\partial V_p}{\partial \omega} \right)^2 \right) ,
\]

so a nonlinear PDE of the 2nd order in two variables (the spherical coordinates \( \theta, \omega \)), obtained by throughout neglecting the \( V_r \) components \( (V_{kr}, V_{\theta r}, V_{\varphi r}) \), so being an approximate PDE, though being an exact one when it is written on the isentropic surfaces \( (V, \Omega) \zeta = \zeta_0 \) (containing streamlines & vortex lines) only, becoming now identical to the general ODE of the velocity quasi-potential; \( V_r \) – the rotational part of \( V \); \( V_{i}(\phi) = V_{ri}(\phi) \) (the radial component of the velocity on the conical isentropic sheet “i”). The quasi-potential is: \( \Phi_i(\phi) = R_i V_i \), also having: \( R_i' / R_i = V_i / V_i \) (see [11]), getting by integration: \( R_i = R_i(\phi) \), representing the polar equation of all flow streamlines contained in the isentropic surface “i” (all being homothetic space curves, their family being given by the polar equation: \( R_i = R_0 R_i(\phi) \), for various values of the positive constant \( R_0 \), with \( R_0(\phi) \) – a non-dimensional quantity depending on \( \phi \) only). The generalized spherical coordinate \( \phi \) is defined by:

\[
\phi = \int (d\theta^2 + \sin^2 \theta d\omega^2)^{1/2} = \int \sin \theta \left[ (d\ln[\tan(\theta/2)])^2 + d\omega^2 \right]^{1/2} \quad \text{(8.1)}
\]

– the relation with the classical spherical coordinates, due to flow’s kinematics (see [11]):

\[
\phi^2 = \theta^2 + \sin^2 \theta \cdot \omega^2 . \quad \text{(8.2)}
\]

In the smart intrinsic curvilinear coordinate system:

\[
\beta = \int (d\phi^2 + f^2(\phi) d\chi^2)^{1/2} ,
\]

which on \( \chi = \chi_{0i} = \text{const.}_i \) becomes: \( \beta = \phi_i \). The general ODE above is similar to the classical Taylor–Maccoll one ([12], [13]) for an axisymmetric conical (supersonic) irrotational flow, this one being a particular case obtained for:

\[
\phi = \theta \quad \text{and} \quad f(\phi) = f(\theta) = \sin \theta , \quad \text{and so:} \quad |\ln|f(\phi)||' = |\ln|\sin \theta||' = \cot \theta , \quad \text{the equation becoming:}
\]
\[(\gamma - 1) \left\{ V'' + \cot \theta \cdot V' + 2V \right\} (W^2 - V^2 - V'^2) - 2(V'' + V)V'^2 = 0, \]

and the sliding condition [11]:
\[ R'V = RV, \text{ with: } ' = d/d\theta \text{ and } '' = d^2/d\theta^2, \]
and the solution \[ V(\theta) = V_R \] – valid throughout (the velocity quasi-potential \( \Phi_i(\varphi) \) coincides in this case with the velocity potential \( \Phi(\theta) \)). Along the “vector radii” half-straight lines one gets a 2-D Laplace’s PDE, with the solution \( \Phi_i/R = V_i = V_{R_i} = \text{const.}_i(\theta) \). Through any flow’s point an “ij” line is passing (the intersection of two conical surfaces: the isentropic “i” sheet \( \chi = \chi_{0i} = \text{const.}_i \) and the isotachic & isothermal “ij” one \( \varphi = \varphi_{0j} = \text{const.}_j \)). This means the searched “ij” line is a half-straight line passing through the flow apex and the considered flow’s point. Along any such a half-straight line the searched for velocity quasi-potential \( \Phi_{ij}(\varphi, \chi) \) becomes a harmonic function (the general non-linear Steichen’s PDE of the 2nd order becoming a 2-D Laplace’s PDE:
\[ 0 = \Delta \Phi_{ij} = 0 – \text{with the solution: } \Phi_{ij}/R = V_{ij} = V_{R_{ij}} = \text{const.}_{ij}(\varphi, \chi). \]

The annulment of the radial component of the acceleration vector, \( a_R = 0 \) (the first equation of motion: \( R - R\varphi^2 = 0 \)), is identical to the “sliding condition”, expressing the condition that the velocity vector \( V \) be tangent to a certain streamline of the conical flow – the scalar product \( V \cdot n = 0 \), where \( n \) is the unit vector of the normal to the streamline, having the components \( R_1, R_2, R_3 \), respectively.

It can be written in the forms:
\[ R'V = RV(= \Phi), \text{ or } R'/R = V/V'; \quad ' = d/d\varphi \text{ (just the streamline ODE), leading to:} \]
\[ \frac{d\ln R}{d\varphi} \frac{dR}{d\Phi_i} = 1, \text{ or: } \frac{d\ln R}{d\varphi} \frac{d\ln |V_{R_i}|}{d\varphi} = 1. \]

In fact, the “velocity quasi-potential” function \( \Phi_i \) must also be a conical one, which means to be in the form:
\[ \Phi_i(R(\varphi, \chi_{0i})) = \Phi_i(R(\varphi)) = R^n \cdot V(\varphi, \chi_{0i}) = R^n \cdot V_i(\varphi), \text{ this being a conical scalar “quasi-potential” of the } n \text{-th order } (n \in \mathbb{Z}); \text{ let choose its simplest form, that for the first order } (n = 1), \text{ having also a physical significance:} \]
\[ \Phi_i(R(\varphi, \chi_{0i})) = \Phi_i(R(\varphi)) = R \cdot V(\varphi, \chi_{0i}) = R \cdot V_i(\varphi); \quad \Phi_i(\varphi) = R \cdot V_i(\varphi); \quad ' = d/d\varphi \]
\[ \text{the trivial system (product and sum of the logarithmic derivatives of } R \text{ and } |V_{R_i}| \text{ with respect to the generalized spherical coordinate } \varphi), \text{ its solution and the analysis on the existence domains for the “(quasi-)potential” } \Phi_i(\varphi), \text{ both from the mathematical viewpoint and the physical one (also including fig. 1, here reproduced as fig. 2) – all from subsection 2.1 in [11], become valid again, this time for any 3-D conical flow (both potential and rotational). So, for an axisymmetric conical flow around a circular cone this domain is: } \beta \in (\beta_0, \beta_L) \text{ with } \beta_0 = \pi/2 - \theta_i \text{ and } \beta_L = \pi/2 - \theta_e, \text{ where: } \sin^2 \theta_e \approx 1/M^2 + [(y+1)/2] \sin^2 \theta_e \text{ (see [14], [15]); } \theta_i, \theta_e \text{ are the solid cone and shock wave half-angles, resp., meaning that } \sin \mu = 1/M, \sin \nu = [(y+1)/2] \sin \theta_e \text{ and } \sin \theta_e (all < 1) \text{ are usually quasi-Pythagorean numbers (} \sin^2 \mu + \sin^2 \nu \approx \sin^2 \theta_e \text{), with } \sin^2 \theta_e < [2/(y+1)] \cdot (1 - 1/M^2)) \text{; } M \text{ is the emergent supersonic stream Mach number, with: } 1/M = \sin \mu. \text{ If is known or is given the conical flow velocity (quasi-)potential } \Phi_i(\beta), \text{ the system above admits the solution:} \]
\[ \frac{d\ln R}{d\beta} = \ln \sqrt{|\Phi_i|} \left( \frac{d\ln |\Phi_i|}{d\beta} \right)^2 - 1 \quad ; \]
\[ \frac{d\ln |V_{R_i}|}{d\beta} = \ln \sqrt{|\Phi_i|} \left( \frac{d\ln |\Phi_i|}{d\beta} \right)^2 - 1, \]
ODEs which can be easily integrated, so being found the solution \((\ln R, \ln |V_{Ri}|)\), and further the \((R, |V_{Ri}|)\) one. We introduce now the hyperbolic functions:

\[
\cosh u = \frac{d\ln \sqrt{|\Phi_i|}}{d\beta}; \quad \sinh u = \sqrt{\left(\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}\right)^2 - 1},
\]

so expressing the solution in the simpler form below:

\[
\frac{d\ln R}{d\beta} = e^{-u} = e^{-\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}}; \quad \ln R = \int e^{-\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}} d\beta;
\]
\[
\frac{d\ln |V_{Ri}|}{d\beta} = e^{u} = e^{\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}}; \quad \ln |V_{Ri}| = \int e^{\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}} d\beta;
\]
\[
\ln \frac{R}{R_0} = \int_{\beta_0}^{\beta} e^{-\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}} d\beta; \quad R = R_0 e^{\beta_0 e^{-\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}} d\beta};
\]
\[
\ln \frac{|V_{Ri}|}{|V_{Ri_0}|} = \int_{\beta_0}^{\beta} e^{\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}} d\beta; \quad |V_{Ri}| = |V_{Ri_0}| e^{\beta_0 e^{\text{argch}\frac{d\ln \sqrt{|\Phi_i|}}{d\beta}} d\beta}.
\]

The existence conditions of the roots in first solution give: \(|\Phi_i| \leq \Phi_i 0 e^{-2\beta}\) and \(|\Phi_i| \geq \Phi_i 0 e^{2\beta}\), with \(\Phi_i 0 > 0\), relations which establish the existence domains of the velocity potential \(\Phi_i(\beta)\) to obtain isentropic 2-D (plane, axisymmetric and general 3-D) conical flows. It can be noticed that for \(\beta > 0\)
can not exist isentropic 2-D conical flows in the domains between the two exponential curves and respectively, between their symmetric ones with respect to the \( \beta \) axis, as one can see in fig. 2. The equations of motion are getting the special forms (2.1.14), (2.2.1) and (2.2.2) in [11]. The current cones \( (C_2) \): \( \chi = 1/C_2 \) represent the isentropic sheets \( \chi = c(S - c_0) = c(S_0 - c_0) = \chi_0 = 1/C_2 \), in which is analyzed the conical motion, these sheets being a particular case of D. Bernoulli surfaces, namely for isoenergetic flows (rigid surfaces in the fluid flow); the velocity \( V \) deriving from a conical quasi-potential of the first order, on each “i” such a sheet \( \chi = \chi_0 \) having: \( V_i = \nabla \Phi_i; \Phi_i = R \cdot V_i(\phi) \) - a conical scalar quasi-potential, specific to the respective “i” sheet, the general flow being rotational \( \Omega \neq 0 \), but with \( \Omega_x = 0 \), so that \( \Omega = k_R \Omega_R + k_\phi \Omega_\phi \) – also see section 1 in [8].

In the classical spherical coordinate system \( (R, \theta, \phi) \) and in the smart intrinsic generalized spherical one \( (R, \phi, \chi) \), respectively, the first equation of motion for a certain 3-D conical flow is:

\[
a_R = R - R\beta^2 = R - R\phi^2 = (\partial p / \partial R)/\rho = 0, \quad \text{and} \quad a_R = R - R\phi^2 - R\phi^2(\phi) \cdot \chi^2 = R - R\beta^2 = R - R\phi^2 = 0; \quad (\chi = 0).
\]

In the case of a plane conical flow: \( \theta = \pi/2 \) (the plane of motion); \( \Rightarrow \sin \theta = 1 \) and: \( \dot{\theta} = 0; \quad \phi = \dot{\phi}. \)

In the case of an axisymmetric conical flow, the streamline is contained in a meridian plane (the phenomenon is independent of \( \omega \)): \( \omega = \omega_0 = \text{const.}; \Rightarrow \quad \omega_0 = \dot{\phi} = \theta. \)

In the case of a general 3-D conical flow, using the notation (10), one obtains the relation (9). There also are helicoidal conical flows (see [11]) – \( \theta \) and \( \omega \) bound by a certain DE.

For all 2-D conical flows the first equation of motion takes the forms:

\[
R - R\beta^2 = 0, \quad \text{or: } (\text{dln} R / \text{d} \beta)(\text{dln} R / \text{d} \beta) = 1
\]

– the product of a pair of logarithmic derivatives with respect to \( \beta \) equal to the unity (\( \beta \) is the angular coordinate:

\( \beta = \theta \) and \( \beta = \pi/2 - \theta \) – for the axisymmetric conical flows (Busemann–Taylor–Maccoll inside and outside a circular cone);

\( \beta = \phi \) – for any 3-D conical flow in the intrinsic system, thus always having:

\[
\arctan(\text{dln} R / \text{d} \beta) + \arctan(\text{dln} R / \text{d} \beta) = \pi/2; \quad \text{(the curves ln} R \text{ and ln} |R| \text{ are symmetrically inclined with respect to the first bisectrix of } (\beta, \Phi) \text{ axes for any value of } \beta, \text{ like a pair of characteristic lines with respect to the velocity } V \text{ direction, in both physical and hodographic plane).}
\]

A simple qualitative example of 3-D conical flow is given in fig. 3 (also see fig. 1 in [2] – [4]). The function \( F(x_1) = \ln \phi^{1/2} + i \tan^{-1}(\chi) \quad \chi \in (-\infty, \infty), \)

represents the complex plane \( x_1 \) on an infinite strip in a plane \( X = Y + i Z \).

Inside this strip, directed on the \( Y \) axis, a parallel and uniform field is obtained. The inverse of the complex function \( F(x_1) \) is multiform. Along the strophoid

\[
\hat{y}^2 + (\hat{z} + \hat{t} - \hat{r})^2 / (\hat{z} + \hat{t} - \hat{r})^2 = 0
\]

(passing through (a), and with (b) as nodal point) \( \chi = 0 \) and so we get \( \hat{Z} = 0 \) – the \( Y \) axis of the complex \( X \) plane. Above strophoid’s branches and inside its loop \( \chi < 0; \hat{Z} < 0 \) (the inferior half-strip in the \( X \) plane). The parallel and uniform flow inside the infinite strip \( \pm \pi/2 \) represents both external and internal flow (superposed). So, the strip’s ends \( \pm \infty \) correspond to (a) and (b) in the internal flow, and to \( x_1 = -i \infty \) (a “slender body”; no shock wave), also giving the relations between the “generalized spherical” smart intrinsic coordinates \( (R, \Phi, \chi) \) and the Cartesian ones; the flow is given by two semi-infinite line sources of opposite senses along cone’s axis (a) and back (b), replacing the solid cone effect, with \( Q(b) = -2Q(a), Q(l) \) being the respective line source’s volumetric flow rate; (a), (b) - the traces of nodal half-straight lines: (a) - for the internal flow only; (b) – for both external and internal flow (in the real compressible flow known as Ferri’s singularity for entropy); (n) - nadir (the lowermost point of the cross section) - the trace of the half-straight line of flow’s saddle points;
Richard Selescu

Fig. 3 Simple qualitative example of the “generalized spherical” smart intrinsic coordinate surfaces for the case of a circular cone at small angle of attack (cross section) – the “incompressible” approximation

3.a. the conical isentropic sheets $\chi = \chi_0_i = c(S_0_i - c_0) = 1/C_2$ (a smart intrinsic coordinate tied to $S_0_i$ - the local specific entropy value), having as remarkable directrices: the oz axis and a circle (the solid cone trace) centered on it (both for $C_2 = 0$), and a right strophoid ($\chi = 0$, passing through (a) and (b), having as asymptote a parallel to the oy axis, of equation: $z = 2r - 1$), centered on the oz axis too (for $1/C_2 = 0$); the constants $c$, $S_0_i, c_0$ are $> 0$;

3.b. the conical sheets $\phi = \phi_0_j = C_1$ (another smart intrinsic coordinate), orthogonal to the conical isentropic ones $\chi = \chi_0_i = c(S_0_i - c_0)$: $\nabla \bar{Y}(\bar{y}, \bar{z}) - \nabla \bar{Z}(\bar{y}, \bar{z}) = \frac{\partial \bar{Y}}{\partial \bar{y}} \frac{\partial \bar{Z}}{\partial \bar{y}} + \frac{\partial \bar{Y}}{\partial \bar{z}} \frac{\partial \bar{Z}}{\partial \bar{z}} = 0$, and having as remarkable directrices: a Pascal’s limaon and the circle at infinity (both for $C_1 = 0$ and centered on the oz axis).

9 Other possible application: a supersonic axisymmetric rotational flow (quasi-conical), confluent to a potential one (tronconical)

In a series of papers (see [16] – [19]) a new shock-free axisymmetric configuration inspired by a Schlieren picture (fig. 260 – a forebody in supersonic flow with isentropic compression – achieved by isentropic way (smoothly) and not by shock) from the flow visualizations album [20] was proposed and studied. This flow was called tronconical, due to the fact that the simple compression waves are co-axial truncated cones having as a common basis (directrix) the “foci” circle, along which they focus into an axisymmetric shock wave (with curved meridian line, and thus with variable intensity) – see fig. 3.a, and various types of air intake were imagined: frontal, annular, frontal-annular, etc. (see figs. 3 – 7 in [18]), all with dynamic compression.
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The tronconical potential flow is associated (confluent) to a rotational flow 2 - 2' treated in [21], around an axisymmetric body without incidence, consisting of a cylinder of radius \( r_0 \) and a truncated cone of half-angle \( \tau_c \) (see fig. 3.a), with a bow shock wave, assumed to be attached at the intersection circle \( CC' \) of the two axisymmetric surfaces. In order to determine the solution (the velocity field and implicitly that of static pressure, the shape of the shock wave and the entropy gradient), the author develops an extremely interesting analytic perturbations method, the disturbance affecting not as usual the incident parallel and uniform supersonic flow of Mach number \( M_1 \) from the upstream, but a potential flow very close to that rotational real, called by him a quasi-conical motion. Later, this problem was resumed in [22], also considering a supersonic combustion wave in the presence of a truncated cone, giving an analytic solution. The confluent flows are related to other simpler supersonic potential flows (the conical axisymmetric one and the plane one with shock wave or Prandtl–Meyer isentropic compression obtained as particular cases; for the last one see fig. 227 from [20]). All tronconical waves (truncated cones) are envelope surfaces of the Mach cones (the local simple waves) generated by the points of the given axisymmetric body surface, all situated in the same cross section plane.

In the toroidal (tronconical) coordinate system \((r_0, R, \theta)\) the PDE of the velocity potential can be set in the forms

\[
\left(1 - \frac{V'^2}{a^2}\right) \left(V'' + V\right) + \frac{1}{1 + \frac{r_0}{R \sin \theta}} \left(\cot \theta \cdot V' + V\right) = 0,
\]

(with \( R \sin \theta \neq -r_0 \)), therefore a nonlinear ODE, \( V \) being the radial component of the velocity (\( V_R = V \)), and \( a \) the local speed of sound, with \( \' = d/d\theta \) and \( \'' = d^2/d\theta^2 \), or

\[
\left(1 + \frac{r_0}{R \sin \theta}\right) \left(1 - \frac{V'^2}{a^2}\right) \left(V'' + V\right) + \cot \theta \cdot V' + V = 0,
\]

The tronconical (shock-free compression) flow inside an axisymmetric supersonic inlet diffuser with

![Fig. 4](image-url)

(a) outer and respectively (b) inner compression (frontal and annular air intake): a) 1 – homentropic cylindrical core of radius \( r_0 \) with tronconical flow – flow with complete (until \( M_2 = 1 \)) isentropic compression; 2 – 2’ – rotational ring – a tronconical flow too, but with expansion, downstream of an attached bow shock wave, weakened and canceled by the tronconical expansion waves (whose
axisymmetric fan was not represented) originating on the “foci” circle CC’, the same for both flows; b) 1 – fuselage (cylinder-cone body); 2 – 2’ – homentropic ring with tronconical flow inside a specially shaped annular channel; 3 – 3’ – infinite ring with non-perturbed supersonic flow.

Analyzing both equations one can see that in these cases the ODE of the velocity potential gets typical forms. Thus, for various values of the cylinder radius \( r_0 \), we have:

\[
r_0 → ∞ ; \quad [1 – (V'^2/a^2)](V'' + V) = 0, \text{ hence } V' = a,
\]

representing the plane flow with Prandtl–Meyer isentropic compression;

\[
r_0 = 0; \quad V'' + \cot θ \cdot V' + 2V = (V'^2/a^2)(V'' + V),
\]

representing the conical axisymmetric flow with isentropic compression inside Busemann’s nozzle (upstream of a conical shock wave with the same tip as the conical simple waves fan), as well as that downstream of an attached conical shock wave (between the shock wave and an infinite solid cone).

In the trivial case of a parallel and uniform supersonic stream (\( R \sin θ = kr_0 \), with \( k ≥ 1 \)) the velocity potential ODE becomes:

\[
\left( 1 + \frac{1}{k} \right) \left( 1 – \frac{V'^2}{a^2} \right) (V'' + V) + \cot θ \cdot V' + V = 0.
\]

Both Prandtl–Meyer (see [23] – [27]) and Taylor–MacColl (see [12], [13]) ODEs (which describe the plane and respectively axisymmetric conical flows) are obtained as simple limit cases of the general ODE of the velocity potential for tronconical flows. From this equation it can be noticed that, unlike the conical flows cases, when the equation has an unknown function of a single variable – \( V(θ) \), in the tronconical flow case, the equation has the same unknown function \( V \), but depending on two variables – \( θ \) and \( R \). In this equation intervening only the function derivatives with respect to \( θ \), the dependence on \( R \) can be considered weaker or even zero (this being called by the author the tronconical approximation: [16] – [18]). Setting (also see fig. 1 from [20]):

\[
\theta = α; \quad R = r; \quad r_0 = b; \quad \frac{1}{1 + r_0/(R \sin θ)} = \frac{R \sin θ}{r_0 + R \sin θ} = \frac{r \sin α}{b + r \sin α} = m;
\]

\[
V = φ(r, α); \quad V' = φ_α; \quad V'' = φ_αα; \quad a = a_1.
\]

one obtains for the general differential equation the form:

\[
\left( 1 – \frac{φ^2_α}{a^2_1} \right) φ_αα + m φ_α \cot α + \left( 1 + m – \frac{φ^2_α}{a^2_1} \right) φ = 0,
\]

identical with Eq. (17) from [21] (but the last one having different boundary conditions), describing an axisymmetric potential flow – a component (namely just the quasi-conical one) of the rotational general supersonic one downstream of an attached axisymmetric bow shock wave. The boundary conditions above are given by the relations:

\[
α = α_0; \quad (φ_α)_0 = 0, \quad \text{ where } α_0 = τ_c \text{ (see fig. 3)} \quad (18 \ a) \text{ in [21]}
\]

\[
α = α_s; \quad φ_s = \cos α_s; \quad (φ_α)_s = – \left( k + \frac{1-k}{K_s^2} \right) \sin α_s \quad (18 \ b) \text{ in [21]}
\]

\( α_s \) is the half-angle of a certain tronconical expansion wave (unknown, function of \( r \)), weakening the bow shock; \( k = λ^2 = (γ − 1)/(γ + 1); \ K_s = M_1 \sin α_0; \ K_s = M_1 \sin α_s. \)

The quantity \( a_1 \) in Eq. (11) is given by the energy equation:

\[
a_1^2 = a^2 = (γ − 1)(W^2 − V'^2 − V^2)/2.
\]
The solution was obtained by analytic way, the general integral of Eq. (11) being given in [21] by the
relations (21), (22), (24 a) and (24 b), for the linearized equation:
\[ \varphi = (C_0 I + D_0) \cos \alpha, \]
where \( C_0 \) and \( D_0 \) are functions of the variable \( r \) only, using the notations below for \( I, I_s \) and \( E(\alpha) \):
\[ C_0 = \frac{\sin 2 \alpha_0}{2 + I_s \sin 2 \alpha_0}, \quad D_0 = 1 - C_0 I_s, \]
\[ I(\alpha, r) = \int_{\alpha_0}^{\alpha} \frac{b + r \sin \alpha}{b + r \sin \alpha_0} \cos \tau \; d\tau = E(\alpha) - E(\alpha_0) - \frac{2 r^2}{b^2} - r^2 \left(1 + \frac{r}{b} \sin \alpha_0 \right) \int_0^{\alpha_0} \frac{b^2 \; d\tau}{(b^2 - r^2)^2}; \quad I_s = I(\alpha_s, r); \]
\[ E(\alpha) = \frac{b + r \sin \alpha_0}{b + r \sin \alpha} \left( \tan \alpha - \frac{2 r^2 b + rz}{b^2 - r^2 + z^2} \right); \quad z = \tan \frac{\alpha}{2}; \]
\[ \tau \text{ being an integration variable. The last integral in (24 a) takes various forms, function of the ratio } \frac{r}{b} \text{ – also see the relations (14) and (15) from [22].} \]
\[ \text{The entire (complete) nonlinear equation of the flow in the downstream of an attached axisymmetric shock wave} \]
\[ q_{\alpha \alpha} + m q_{\alpha} \cot \alpha + (1 + m) \varphi = F(r, \alpha) + N(r, \alpha), \]
\[ \text{having in its right-hand side the nonlinear terms } F \text{ and } N \]
\[ F(r, \alpha) = \frac{\varphi_{\alpha}^2}{a_1}, \quad \varphi_{\alpha \alpha} + \varphi, \quad \text{(19) in [21]} \]
\[ \text{(just the nonlinear term of Eq. (11), neglected until now);} \]
\[ N(r, \alpha) = \left( \frac{\varphi_{\alpha}^2}{a_1^2} - m \right) r \varphi_\alpha + 2 \frac{\varphi_{\alpha} \varphi_{\alpha \alpha}}{a_1^2} r \varphi_{\alpha \alpha} - (2 r \varphi_\alpha + r^2 \varphi_{\alpha \alpha}) \left(1 - \frac{\varphi_{\alpha}^2}{a_1^2} \right), \]
\[ \text{(16) in [21]} \]
\[ \text{has the following general integral:} \]
\[ \varphi = [(C + H) I + (D - L)] \cos \alpha; \]
\[ \text{C and } D \text{ are functions of the variable } r \text{ only, and are determined imposing the boundary conditions, like } \alpha_s; \]
\[ H = \int_{\alpha_0}^{\alpha} \frac{[b + r \sin \tau] [F(r, \tau) + N(r, \tau)]}{b + r \sin \alpha_0} \cos \tau \; d\tau, \quad \text{and } L = \int_{\alpha_0}^{\alpha} \frac{[b + r \sin \tau] [F(r, \tau) + N(r, \tau)]}{b + r \sin \alpha_0} \cos \tau \; d\tau, \]
\[ \text{representing the relations (33 a) and (33 b) in [21], respectively.} \]
The integral \( I \) is given by the relations (24 a), (24 b) and respectively (32), (33 a) and (33 b), for the
total nonlinear equation. In the last two relations, we must consider for the second nonlinear term, \( N(r, \alpha) \), defined by the relation (16) from [21], the condition \( N(r, \alpha) = 0 \), this corresponding to our
tronconical flow case.

An improvement in our problem solution can be obtained by replacing the relation for the main boundary condition: \( \textbf{V} \cdot \textbf{n} = 0 \) with the new (exact) streamline’s equation:
\[ \textbf{R}^{\prime} \textbf{V}^{\prime} = \textbf{RV} (= \Phi), \text{orr}_{\alpha} \varphi_{\alpha} = r(\varphi + r \varphi_\alpha), \]
\[ \text{(9.2)} \]
\[ \text{with: } r_\alpha = dr/d\alpha, \varphi_{\alpha} = \partial \varphi / \partial \alpha \text{ and } \varphi_\alpha = \partial \varphi / \partial r, \]
where, taking into account the definition of the velocity potential – relation (14) in [21], assumed in
the simple new form \( \Phi = r \varphi(r, \alpha) \) (therefore not a tronconical one, \( r \varphi(\alpha) \)), were used for the velocity
components the expressions
\[ V_R = V_r = \partial \Phi / \partial r = \varphi + r \varphi_\alpha; \quad V_\theta = V_\alpha = (1/r) \cdot (\partial \Phi / \partial \alpha) = \varphi_\alpha. \]
On the other hand, in order to replace the expression \( a_1^2 \) in the general PDE of the tronconical flow –
the former ODE (9.1), one must write a new (exact) form of the energy PDE:
\[ a_1^2 = (\gamma - 1) [W^2 - (\varphi + r \varphi_\alpha)^2 - (\varphi_{\alpha})^2]/2. \]
The new solution will be obtained by integrating the system of three PDEs (9.1), (9.2), (9.3) (the subscripts \( \alpha \) and \( r \) having now the significance of partial differentiation with respect to the respective variables), these equations ceasing to still represent the mathematical model of a tronconical flow, but of a new flow, related to that considered in [21]. This new system, by solving of which is obtained the improved solution, has three unknowns: \( \phi(r, \alpha) \), \( a_1(r, \alpha) \), and \( r(\alpha) \) – the streamline (body wall) equation, the functions \( \phi \) and \( a_1 \) depending on \( \alpha \) both directly and via \( r \) as well, so that one can write on a certain streamline: \( d\phi = \phi_\alpha d\alpha + \phi_r r_\alpha d\alpha = (\phi_\alpha + \phi_r r_\alpha) d\alpha \) and respectively: \( da_1 = (a_1_\alpha + a_1 r_\alpha) d\alpha \). It is also to be noticed that in all the cases, the equation of the velocity potential admits the particular (trivial) solution \( V = -U_\infty \cos \theta \) (a parallel and uniform stream). In the case of absence of the special shaped central body (see fig. 3.a), when only the cylinder-cone shell exists, the outer quasi-conical motion is no more associated (confluent) to the inner tronconical flow with isentropic compression, but to a non-tronconical flow with expansion, considered in [28], called by the author Prandtl-Meyer expansion in axisymmetric flow. It must be mentioned that the general equation from the beginning of this section can describe any tronconical flow, not only those with isentropic compression.

Thus, it can be imagined a tronconical flow with expansion in an axisymmetric nozzle similar to Busemann’s one, but having inside a central cylinder-cone body and a pre-determined shape of the wall meridian line, corresponding to a strictly calculated design Mach number of the downstream parallel and uniform supersonic flow (also see fig. 3.b and fig. 173 from [29]). This flow can be easily obtained by a simple reversal of the flow from figure 3.b and extends itself only in the supersonic range. This is possible, because the inner flow is entirely shock-free (and thus isentropic).

In the same manner, starting from the tronconical flow with isentropic compression represented in fig. 3.a one can obtain the tronconical flow with expansion given in fig. 4.1.a from [30]. Besides these cases, there also is that of the tronconical flow with expansion in the second region of the mixed flow (isentropic compression – expansion) around a specially shaped axisymmetric forebody, immediately followed by a cylinder (see fig. 7 in [18], here reproduced as fig. 4). This expansion flow succeeds to a tronconical flow with isentropic compression in the first region of the above flow and takes place just around the circular edge (having in the meridian plane the trace B) of the intersection of the two bodies, this edge being just the foci circle of the tronconical flow with expansion. Besides these methods, another way to obtain tronconical flows with expansion consists in the generalization of their corresponding axisymmetric conical flows (see [31] – [34]), instead of the conical flow’s vertex appearing now the tronconical flow’s circle of foci.

The new flow’s velocity potential equation (valid for tronconical flows with both compression and expansion) was established as a first step in finding the exact solution and a streamlining method for the axisymmetric bodies and channels, useful for applications, was imagined. The confluent supersonic flows around an axisymmetric specially shaped configuration, consisting in a central forebody followed in the downstream by a co-axial inlet diffuser (annular air intake) In order to unify the methods of solution for the PDEs of the confluent flows from figs. 3.a or 4 (potential – internal and rotational – external, both having a strong tronconical character) without dividing the external flow into a potential (quasi-conical) one, governed by the same PDE of the velocity potential (11), and a rotational perturbation, and then adding (superposing) these terms of solution, we can apply the new 2-D velocity quasi-potential theory on the whole axisymmetric flow’s isentropic surfaces. Thus, even the external rotational flow can be treated as a potential one, if we can determine the equation of the axisymmetric isentropic surfaces, starting from the already known streamline’s Eq. (12), for a given velocity field \( r\phi(r, \alpha) \).
10 First approach for extension of the new quasi-potential model to the rotational flow of a viscous Newtonian compressible fluid

Let start from the vector general form of the motion equation for a viscous Newtonian fluid flow (Navier–Stokes):

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla \left( \mathbf{V}^2 \right) + \mathbf{\Omega} \times \mathbf{V} = \mathbf{f} - \frac{\nabla p}{\rho} + \frac{\mu_1}{\rho} \Delta \mathbf{V} + \left( \frac{\mu_2}{\rho} + \frac{\mu_1}{3\rho} \right) \nabla (\nabla \mathbf{V})$$

(the Helmholtz (see [35])–Gromeko–Lamb form, binding the acceleration and force density terms of a fluid particle); \(\mathbf{f}\) – the mass force density (conservative – a gradient): \(\mathbf{f} = \nabla (–gz) = –\nabla (gz)\); \(g\) – the acceleration of gravity; \(z\) – the geometrical height (height of the considered point above a reference horizontal plane \(xOy\)); \(\mu_1\) – the dynamic viscosity of the fluid or the coefficient of internal friction; \(\mu_1/\rho\) – kinematic viscosity of the fluid; \(\mu_2\) – second, or bulk, viscosity (\(\mu_1, \mu_2\) – assumed constant).

For a steady motion and respectively for gases, we have: \(\partial \mathbf{V}/\partial t = 0\) and \(\mathbf{f} = 0\), thus remaining:

$$\nabla \left( \mathbf{V}^2 \right) + \mathbf{\Omega} \times \mathbf{V} = -\frac{1}{\rho} \left[ \nabla p - \mu_1 \Delta \mathbf{V} - \left( \mu_2 + \frac{\mu_1}{3} \right) \nabla (\nabla \mathbf{V}) \right].$$

Searching for a 2-D velocity quasi-potential \(i (\xi, \eta)\) (therefore in a new smart intrinsic coordinate system \(O\xi\eta\zeta\)), we will perform first a scalar multiplication of this equation by a special virtual elementary displacement \(d\mathbf{R} \in (\mathbf{V}, \mathbf{\Omega})\) plane (see section 2), so obtaining some surfaces similar to the isentropic ones for the inviscid compressible fluid flow case, envelope sheets of the \((\mathbf{V}, \mathbf{\Omega})\) planes above. Over these surfaces we can write the analytic expression:

$$\mathbf{\Omega} = \nabla \times \mathbf{V} = \frac{1}{h_\xi h_\eta h_\zeta} \begin{vmatrix} h_\eta k_\zeta & h_\eta k_\eta & h_\eta k_\xi \\ \frac{\partial}{\partial \zeta} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial \xi} \\ h_\xi V_\zeta & h_\eta V_\eta & h_\zeta V_\xi \end{vmatrix} = k_\xi \Omega_\zeta + k_\eta \Omega_\eta + k_\zeta \Omega_\xi,$$

with \(V_\zeta = h_\xi \dot{\xi}; V_\eta = h_\eta \dot{\eta}; V_\xi = h_\zeta \dot{\zeta}\) = 0, where: \(k_i\) – a 3-D basis; \(h_i\) – Lamé’s coefficients; the dotted variables are derivatives with respect to the time \(t\), and so:

$$\Omega_\xi = \frac{1}{h_\xi h_\eta} \left[ \frac{\partial (h_\eta V_\eta)}{\partial \xi} - \frac{\partial (h_\zeta V_\zeta)}{\partial \eta} \right] = 0, \text{ or } \frac{\partial (h_\eta V_\eta)}{\partial \xi} = \frac{\partial (h_\zeta V_\zeta)}{\partial \eta} = 0.$$
Let us introduce a scalar function $\Phi_t(M) = \Phi_t(\xi, \eta, \zeta)$, also called by the author a 2-D velocity “quasi-potential”, whose partial derivatives along the directions of the elementary orthogonal arcs $h_\xi d\xi$ and $h_\eta d\eta$ on the “$t''$ $(V, \Omega)$ sheet ($\zeta = \zeta_0$) are just the components $V_{\xi_i}$ and $V_{\eta_i}$ of the velocity vector $(V_{\zeta} = 0)$. Let us still define $\lambda$ and $\mu$ as being two orthogonal arc lengths (also see sections 3, 5 – 7), so that: $d\lambda = h_\xi d\xi$ and $d\mu = h_\eta d\eta$, $(dR = c_1 V + c_2 \Omega = k_\xi h_\xi d\xi + k_\eta h_\eta d\eta = k_\xi d\lambda + k_\eta d\mu)$, on these “$t''$” surfaces having:

$$V_{\xi_i} = \frac{1}{h_\xi} \frac{\partial \Phi_1}{\partial \xi} ; \ V_{\eta_i} = \frac{1}{h_\eta} \frac{\partial \Phi_1}{\partial \eta} ; \ V_{\zeta_i} = \frac{1}{h_\zeta} \frac{\partial \Phi_2}{\partial \zeta} = 0; \ \Rightarrow \ h_\xi V_{\xi_j} = \frac{\partial \Phi_1}{\partial \xi} ; \ h_\eta V_{\eta_j} = \frac{\partial \Phi_1}{\partial \eta} ; \ h_\zeta V_{\zeta_j} = \frac{\partial \Phi_2}{\partial \zeta} ; \ \frac{\partial^2 \Phi_1}{\partial \xi \partial \eta} = \frac{\partial^2 \Phi_1}{\partial \eta \partial \xi} = \frac{\partial^2 \Phi_2}{\partial \zeta \partial \xi} = \frac{\partial^2 \Phi_2}{\partial \zeta \partial \eta} .$$ \hspace{2cm} (10.1)

So the relation $\Omega_{\xi_i} = 0$ leads to: $\partial^2 \Phi_1/\partial \xi \partial \eta - \partial^2 \Phi_1/\partial \eta \partial \xi = 0$, this representing just Schwarz’ theorem for the functions of two variables (the so-called theorem of “the equality of the mixed derivatives of the second order”, they differing as to the order of differentiation only).

This relation proves that $\Omega_{\xi_i} = 0$ and the existence of a 2-D velocity “quasi-potential” function $\Phi_t$ so that:

$$V_{\xi_i} = V_{\lambda i} = \frac{\partial \Phi_1}{\partial \lambda} ; \ V_{\eta_i} = V_{\mu i} = \frac{\partial \Phi_1}{\partial \mu} \hspace{0.5cm} \text{(also see [2] – [5]).}$$

Applying the second law of thermodynamics, one can write the heat transport equation (called by this author the generalized Crocco’s equation for rot-viscous fluids) as:

$$TdS = (\Omega \times V) \cdot dR + \frac{\mu_1}{\rho} \left[ \left( \frac{\mu_2}{\mu_1} - \frac{2}{3} \right) I_1^2 + 2I_2 \right] dt \Rightarrow \left\{ (\Omega \times V) \cdot V + \frac{\mu_1}{\rho} \left[ \left( \frac{\mu_2}{\mu_1} - \frac{2}{3} \right) I_1^2 + 2I_2 \right] \right\} dt = TdS_1 \hspace{0.5cm} \Omega \neq 0; \mu_1 = \mu_2 = 0 + TdS_2 \hspace{0.5cm} (\Omega = 0; \mu_1 \neq \mu_2 = 0), \hspace{0.5cm} \text{(also see [2], [4], [36]) with the following notations:} dR = V_i dt; \ I_1 = e_i^j = e_i^j = e_{11} + e_{22} + e_{33} = \nabla V, \ I_2 = e^{ij}e_{ij} = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11} - e_{12}^2 - e_{23}^2 - e_{31}^2 \geq 0 .$$

$V_i$ is the virtual velocity vector (along the virtual displacement $dR$); $I_1, I_2$ – first and second invariant of the tensor of the fluid particle deformation rate $e$ ([37], [38]). The first three terms $e_{ij}e_{ij}; j \neq i$ in the $I_2$ expression are given by the linear deformation, while the last three ones $(e_{ij}^2; j \neq i)$ are given by the angular deformation:

$$e_{ij} = \partial V_{xi}/\partial x_i = v_{ji}; e_{ij} = (\partial V_{xj}/\partial x_j + \partial V_{xj}/\partial x_i)/2 = (v_{ij} + v_{ji})/2; e_{ij} = (\partial V_{xj}/\partial x_i + \partial V_{xi}/\partial x_j)/2 = (v_{ji} + v_{ij})/2 = e_{ij} .$$

The new equation of the isentropic surfaces (dS = 0, needed to assure that $a^2 = (dp/d\rho)_{S=S_0}$ in Steichen’s PDE) is:

$$T(dS_1 + dS_2) = (\Omega \times V) \cdot dR + \frac{\mu_1}{\rho} \left[ \left( \frac{\mu_2}{\mu_1} - \frac{2}{3} \right) I_1^2 + 2I_2 \right] dt \Rightarrow \left\{ (\Omega \times V) \cdot V + \frac{\mu_1}{\rho} \left[ \left( \frac{\mu_2}{\mu_1} - \frac{2}{3} \right) I_1^2 + 2I_2 \right] \right\} dt = 0 ,$$ \hspace{2cm} (10.2)

having two main particular cases (independent annulment):

1. rotational flow of an inviscid fluid ($\mu_1 = \mu_2 = 0$), in the expression above missing the second term of the sum $(dS_2)$, the motion equation admitting a D. Bernoulli first integral;
2. irrotational flow $(\Omega = 0)$ of a viscous fluid, in this case missing the first term of the sum $(dS_1)$, obtaining:

1. the annulment of the mixed (scalar triple) product in the left-hand side (coplanarity of three vectors), so getting in the Cartesian system Oxyz the following PDE of $(V, \Omega)$ sheets:
\[
d\mathbf{R} \cdot (\mathbf{\Omega} \times \mathbf{V}) / T = 0 = \left| \frac{1}{T} \begin{vmatrix} dx & dy & dz \\ \Omega_x & \Omega_y & \Omega_z \\ V_x & V_y & V_z \end{vmatrix} \right| , \quad \text{or}: \quad \left| \frac{1}{T} \begin{vmatrix} \frac{\partial V_y}{\partial x} - \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} - \frac{\partial V_y}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \right| = 0 ;
\]

in the smart intrinsic system \( O \xi \eta \zeta \) this equation is \( \zeta = \zeta_{0i} \);

2. the annulment of the second factor of the second product in the left-hand side, \((\mu_2/\mu_1 - 2/3)f_1^2 + 2I_2 \), written in the principal deformation rates \((e_{ii} = e_i; e_{ij} = e_{ji} = 0) \):

\[
(1/T) \cdot [(\mu_2/\mu_1 - 2/3)f_1^2 + 2I_2] = 0 \Rightarrow e_1^2 + e_2^2 + e_3^2 - 2(\mu_2/\mu_1 + 1/3)/(2/3 - \mu_2/\mu_1) \cdot (e_1 e_2 + e_2 e_3 + e_3 e_1) = 0 ;
\]

- an isentropic right-circular cone having as vertex the origin and as axis the straight line: \( e_1 = e_2 = e_3 \) (the diagonal passing through \( O_1 \) of the cube of equation: \( e_1 e_2 e_3 (e_1 - a) (e_2 - a) (e_3 - a) = 0 ; \) \( a > 0; \) \( e_1, e_2, e_3 \in [0, a] \)), in the principal Cartesian system \( O_1 x_1 y_1 z_1 \). Replacing \( e_1, e_2, e_3 \) by \( \partial V_{x1}/\partial x_1, \partial V_{y1}/\partial y_1, \partial V_{z1}/\partial z_1 \), we get the PDE of the isentropic cone, having as unknown functions the velocity \( V \) components on the principal axes \( V_{x1}, V_{y1}, V_{z1} \):

\[
\frac{1}{T} \left( \left( \frac{\partial V_{x1}}{\partial x_1} \right)^2 + \left( \frac{\partial V_{y1}}{\partial y_1} \right)^2 + \left( \frac{\partial V_{z1}}{\partial z_1} \right)^2 - 2 \cdot \frac{\mu_2}{\mu_1} + 2/3 \cdot \frac{\mu_2}{\mu_1} \cdot \left( \frac{\partial V_{x1}}{\partial x_1} \cdot \frac{\partial V_{y1}}{\partial y_1} + \frac{\partial V_{z1}}{\partial z_1} + \frac{\partial V_{x1}}{\partial x_1} \cdot \frac{\partial V_{x1}}{\partial x_1} \right) \right) = 0 .
\]

In all these cases \((dS_0 = 0; i = 1, 2)\) the physical equation is: \( p = K_i \rho_T \), with: \( K_i = p_{0i} / (\rho_{0i})^T \cdot \exp[(S - S_{0i}) / C_v] = p_{0i} / (\rho_{0i})^T > 0 \) (the isentropic constant of the respective particular “1” isentropic virtual surface, \( C_v \) being a constant – the isochoric specific heat of the respective ideal gas; \( p_{0i} \) and \( \rho_{0i} \) are stagnation values). Over the isentropic “1” surface the speed of sound is given by: \( a^2 = (dp/d\rho)_{S=S_0} = \gamma RT \)

### 11 The new PDE of the velocity quasi-potential for the steady flow of a viscous Newtonian fluid and its validity domain

Therefore along the intersection space lines of the particular isentropic surface families \((V, \Omega)_{i1}\) sheets and “\( j \)" circular cones with \( K_j = K_j \), the general rotational flow of a viscous Newtonian compressible fluid is governed by Steichen’s PDE of the 2-D quasi-potential \( \Phi_j(M) = \Phi_j(\xi, \eta, \zeta_0) \) (see sections 3 and 5), so at the same point:

\[
\left[ \frac{1}{h_\xi^2} \frac{\partial^2 \Phi_j}{\partial \xi^2} + \frac{1}{h_\eta^2} \frac{\partial^2 \Phi_j}{\partial \eta^2} + \frac{1}{h_\zeta^2} \frac{\partial^2 \Phi_j}{\partial \zeta^2} \right] \left[ \frac{1}{h_\xi^2} \frac{\partial \ln(h_\xi h_\zeta)}{\partial \xi} \frac{\partial \Phi_j}{\partial \xi} + \frac{1}{h_\eta^2} \frac{\partial \ln(h_\eta h_\zeta)}{\partial \eta} \frac{\partial \Phi_j}{\partial \eta} \right] \left[ \frac{1}{h_\xi^2} \frac{\partial \ln(h_\xi h_\eta)}{\partial \xi} \frac{\partial \Phi_j}{\partial \xi} + \frac{1}{h_\eta^2} \frac{\partial \ln(h_\eta h_\xi)}{\partial \eta} \frac{\partial \Phi_j}{\partial \eta} \right]
\]

\[
\left[ \frac{2}{\gamma - 1} \left( \frac{1}{h_\xi^2} \left( \frac{\partial \Phi_j}{\partial \xi} \right)^2 + \frac{1}{h_\eta^2} \left( \frac{\partial \Phi_j}{\partial \eta} \right)^2 \right) + \frac{2}{\gamma - 1} \frac{\partial \Phi_j}{\partial \eta} \frac{\partial \Phi_j}{\partial \xi} \left( \frac{\partial \Phi_j}{\partial \xi} \right)^2 \right]
\]

\[
+ \left[ \frac{1}{h_\xi^2} \frac{\partial \ln(h_\xi)}{\partial \xi} \frac{\partial \Phi_j}{\partial \xi} + \frac{1}{h_\eta^2} \frac{\partial \ln(h_\eta)}{\partial \eta} \frac{\partial \Phi_j}{\partial \eta} + \frac{1}{h_\zeta^2} \frac{\partial \ln(h_\zeta)}{\partial \zeta} \frac{\partial \Phi_j}{\partial \zeta} \right] \left( \frac{\partial \Phi_j}{\partial \eta} \right)^2 + \frac{1}{h_\xi^2} \frac{\partial \ln(h_\xi)}{\partial \xi} \frac{\partial \Phi_j}{\partial \xi} + \frac{1}{h_\eta^2} \frac{\partial \ln(h_\eta)}{\partial \eta} \frac{\partial \Phi_j}{\partial \eta} + \frac{1}{h_\zeta^2} \frac{\partial \ln(h_\zeta)}{\partial \zeta} \frac{\partial \Phi_j}{\partial \zeta} \right] \left( \frac{\partial \Phi_j}{\partial \eta} \right)^2
\]

\[
(11.1)
\]

But the most general case for Eq. (15) to represent the isentropic surfaces of the general steady rotational flow of a viscous Newtonian compressible fluid corresponds to the existence of a special virtual elementary displacement \( d\mathbf{R} \) to satisfy this equation. So, if both terms of the sum in the left-hand side have the same modulus and sign:
\[(\mathbf{\Omega} \times \mathbf{V}) \cdot d\mathbf{R} = \left| \frac{\mu_1}{\rho} \left[ \frac{\mu_2}{\mu_1} \left( \frac{2}{3} \right) f_1^2 + 2f_2 \right] \right| d\mathbf{R}, \quad \text{and} \]

\[
\text{sign} (\mathbf{\Omega} \times \mathbf{V}) \cdot d\mathbf{R} = \text{sign} \left\{ \frac{\mu_1}{\rho} \left[ \frac{\mu_2}{\mu_1} \left( \frac{2}{3} \right) f_1^2 + 2f_2 \right] \right\} ;
\]

in order to assure the annulment of this sum, one has to change the sign of \(d\mathbf{R}\) (and so the sign of the first term of the sum), obtaining a mutual annulment of both terms, \((d\mathbf{S}_2 = - d\mathbf{S}_1)\), therefore satisfying Eq. (10.1): \(TdS = 0\) (TVS is always \(> 0\), but the sign of \(TdS\) depends on \(d\mathbf{R}\)).

This \(d\mathbf{R}\) describes the searched for “\(i\)” virtual isentropic surface. But over these surfaces, even isentropic, the considered flow does not admit a velocity quasi-potential \(\Phi_i\) (like over the \((\mathbf{V}, \mathbf{\Omega})\) ones in the previous case). In subsection 1.4 of papers [2], [4] the dependence of the gas particle specific entropy \(S\) on the velocity “quasi-potential” \(\Phi\) (the interdependence \(S \leftrightarrow \Phi\)) was established in the form below (see Eq. (5) in [2], [4]), valid assuming that the velocity vector \(\mathbf{V}\) is derived from a velocity potential \(\Phi\) only (also see Eq. (10.1)):

\[
dS = - \frac{1}{\pi \sigma \sigma} \left[ \left( \frac{\partial V}{\partial \sigma} \right)^2 + \left( \frac{\partial V}{\partial \tau} \right)^2 \right] d\nu \quad ; \quad S = \frac{1}{\pi \sigma} \int \frac{d\nu}{\sigma} \ln \left[ W^2 - \left( \frac{\partial \Phi}{\partial \sigma} \right)^2 - \left( \frac{\partial \Phi}{\partial \tau} \right)^2 \right] d\zeta
\]

meaning that: \(V_\times = \frac{\partial \Phi}{\partial \sigma} \div \sigma = \frac{\partial \Phi}{\partial \mu} \div \mu = V_\times = \frac{\partial \Phi}{\partial \nu} = 0\), (see Eq. (10.1)), where \(\lambda\) and \(\mu\) are the orthogonal arc lengths in the new smart intrinsic triorthogonal system \(O_\lambda \mu \nu\) (or \(O_\xi \eta \zeta\)) tied to the \((\mathbf{V}, \mathbf{\Omega})\) sheets, so that Eq. (11.1) of \(\Phi_{ij}\) is the single possible 2-D velocity quasi-potential PDE to be written.

### 12 Model extension to the magnetically inviscid (with no magnetic viscosity) MHD (plasma included); validity of Steichen’s PDE

All this analysis can be extended to the inviscid MHD. The results obtained in studying the local physical phenomena in fluid mechanics (& MHD) in a new smart intrinsic coordinate system were presented in [39], and the last researches performed in this field were given in [40], both as plenary lecture (“Deep insight into the still hidden theory of isoenergetic flow”), representing a real “physiology” of the fluid medium, treating the equations of motion, continuity, flow rate, vorticity (and, by analogy, of “magnetic induction” one in MHD) and velocity potential. The equation of the isentropic surfaces is (two cases):

1. \(TdS = [(\mathbf{\Omega} \times \mathbf{V}) - (1/4\pi \rho) \cdot (\mathbf{j} \times \mathbf{H})] \cdot \mathbf{R} = 0\), for an inviscid fluid (see sections 2 in [2], [39], [40], and 3 in [41]);
2. \(TdS = [(\mathbf{\Omega} \times \mathbf{V}) - (1/4\pi \rho) \cdot (\mathbf{j} \times \mathbf{H})] \cdot \mathbf{V} + \mu_1/\rho \cdot [(\mu_2/\mu_1 - 2/3)f_1^2 + 2f_2] \cdot dt = 0\), for a viscous one (see [2], [40]), but for the case of magnetically inviscid MHD (with no magnetic viscosity).

\(\mathbf{V}\) is the local mean instantaneous velocity of translation of the ionized fluid (plasma) particles (atoms, ions etc.) contained in a local small volume element (defined in subsection 2.1 in [42]):

\[
\mathbf{V} = (\rho_+ \mathbf{V}_+ + \rho_\times \mathbf{V}_\times + \rho_- \mathbf{V}_-)/\rho ,
\]

where: \(\mathbf{V}_+\), \(\mathbf{V}_\times\), \(\mathbf{V}_-\) are the velocities of the components; \(\rho_+, \rho_\times, \rho_-\) are the densities of the components;

\[
\rho_a = n_a m_a; \quad \rho_+ = n_+ m_+; \quad \rho_- = n_- m_-; \quad \rho = n_a m_a + n_+ m_+ + n_- m_- = \rho_a + \rho_+ + \rho_- - \rho \quad \text{- plasma density (analogously to the case of a mixture of components); } m_a \text{ is the mass of a neutral atom}\ (m_a = m_+ + m_-); \quad m_+ \text{ is the mass of the positive ion (a single species); } m_- \text{ is the mass of the}
\]
negative ion or of electron; \( n_a \) is plasma concentration in neutral atoms, and respectively (for a three-component neutral plasma): \( n_+ \), \( n_- \) are the concentrations in positive and negative particles (a single species of cations and anions; in the case of a quasi-neutral plasma: \( n_+ \approx n_- \)), all according to a simplified model proposed by the author.

Therefore the flow (mean) vorticity is given by:

\[
\boldsymbol{\Omega} = \nabla \times \mathbf{V} = \nabla \times \left( \frac{\rho_s \mathbf{V_a} + \rho_+ \mathbf{V}_+ + \rho_- \mathbf{V}_-}{(\rho_a + \rho_+ + \rho_-)} \right);
\]

\( \mathbf{H} \) is the intensity of the local magnetic field, using the same convention of equivalence (in the Gaussian system of units, since for all electro-conducting fluids the magnetic permeability (\( \mu \)) is approximately equal to 1 - see for reference [38], [45] - [47]) to the magnetic induction \( \mathbf{B} \) (as a rule variable in the time \( t \)), with: \( \nabla \times \mathbf{H} = 0 \) (\( \mathbf{H} = \nabla \times \mathbf{W} \) - a solenoidal field). \( \mathbf{j} \) is the density of the conduction electric current (see Maxwell’s 2nd equation in [48]):

\[
\frac{j}{k} = \nabla \times \mathbf{H} - \frac{1}{c} \mathbf{E} = \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} + \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \times \mathbf{H}); \quad k = c/4\pi , \quad \text{or, using the low-frequency Ampère’s law (neglecting the displacement electric current):} \quad \frac{j}{k} = \nabla \times \mathbf{H}; \quad c \text{ is the light speed in a vacuum.}
\]

\( \mathbf{E} \) is the intensity of the local induced electric field (also variable in the time \( t \)).

\( \mathbf{V}_v \) is the virtual velocity of the small electro-conducting fluid particle, given by: \( d\mathbf{R} = \mathbf{V}_v dt \).

\( \mu_1 \) is the dynamic viscosity of the fluid or the coefficient of internal friction; \( \mu_1 / \rho \) is the kinematic viscosity of the fluid.

\( \mu_2 \) is the second, or bulk, viscosity of the fluid, assuming that both \( \mu_1 \) and \( \mu_2 \) are constant (mean values).

\( l_1, l_2 \) are first and second invariant of the tensor of the fluid particle deformation rate (also see section 10).

Analogously to section 11, the validity domain for Steichen’s PDE is along Selescu’s magnetohydrodynamic “ij” vector lines: \( d\mathbf{R} || \$ \), with: \( \$ \equiv (\mathbf{V}, \boldsymbol{\Omega}) \cap (\mathbf{H}, \mathbf{j}) = (\boldsymbol{\Omega} \times \mathbf{V}) \times (\mathbf{j} \times \mathbf{H}) \) (see, e.g.: [2], [41], [44]) in the inviscid MHD.

### 13 Conclusions and remarks

In this paper the expanded form of Steichen’s PDE of the velocity potential in a certain triorthogonal curvilinear coordinate system was established, for both steady and unsteady motion of an inviscid compressible fluid.

Choosing a smart intrinsic system \( O\zeta \eta \zeta \) tied to the isentropic surfaces (\( \mathbf{V}, \boldsymbol{\Omega} \)), some simpler forms of the equation were obtained, even for the rotational flows, using a 2-D velocity “quasi-potential” \( \Phi_i(\zeta, \eta) \). Over the flow’s polytropic surfaces the fluid has a quasi-barotropic behavior. So, for any steady rotational flow of an inviscid compressible fluid one can find a 2-D velocity “quasi-potential” \( \Phi_i(\zeta, \eta) \) satisfying Steichen’s PDE over some rigid virtual “\( i \)" isentropic surfaces, and even more, one can find a similar 2-D velocity “quasi-potential” \( \Phi_{ij}(\zeta, \eta) \) satisfying Laplace’s PDE \( \Delta \Phi_{ij} = 0 \) (so a harmonic function \( \Phi_{ij} \) along some rigid virtual “\( ij \)" isentropic and polytropic space curves (intersection lines of “\( i \)" isentropic surfaces with “\( j \)" polytropic ones) - a quasi-incompressible quasi-potential behavior in a rotational pseudo-flow of a compressible fluid. For any unsteady rotational motion of an inviscid compressible fluid one can find a new 2-D velocity “quasi-potential” \( \Phi_i(\zeta, \eta, t) \) which satisfies Steichen’s PDE over some time-deformable virtual “\( i \)" isentropic surfaces. Simpler forms of all the above equations can be obtained using the orthogonal elementary arc lengths: \( d\lambda = h_{\zeta} d\zeta \) and \( d\mu = h_{\eta} d\eta \) (\( dv = h_c d\zeta = 0 \) - isentropic sheet), without any Lamé’s coefficient at the denominators. So, along the streamlines of a 3-D steady rotational flow of an inviscid compressible fluid the motion is given by an ODE of the 2nd order for the 1-D velocity quasi-potential \( \Phi_i(\zeta) \).
while along the intersection lines of the isentropic virtual “i” surfaces with the isothermal & isotachic (polytropic) virtual “j” surfaces (the flow’s Laplace lines) the motion is described by a Laplace’s PDE for the 2-D velocity quasi-potential $\Phi_{ij}(\xi, \eta)$.

Therefore, for any steady rotational flow of an inviscid compressible fluid one can find a 2-D velocity “quasi-potential” $\Phi_i(\xi, \eta)$ satisfying Steichen’s PDE over some virtual “i” isentropic surfaces, and even more, one can find another 2-D velocity “quasi-potential” $\Phi_{ij}(\xi, \eta)$ satisfying Laplace’s PDE (so a harmonic function $\Phi_{ij}(\xi, \eta)$) along some virtual “ij” isentropic and polytropic space curves (that is the intersection lines of the “i” isentropic surfaces with the “j” polytropic ones), that means an elliptic PDE (its canonical form), irrespective of the pseudo-flow character (subsonic or supersonic). Their direction in space is given by the local Selescu’s isentropic & isotachic elementary virtual displacement vector $(dR_{ij} = k_1 \cdot \nabla S_{ij} \times \nabla V_{ij})$ mainly. Through any point of the rotational flow of an inviscid compressible fluid always passes such an incompressible “Laplace” “ij” line (isentropic & isotachic), except for the case when both surfaces are identical, thus appearing an undetermined solution. On the other hand, choosing the smart intrinsic coordinate $\xi$ just along flow’s streamlines (assumed as being known), Steichen’s PDE becomes an ODE of the 2nd order, therefore simpler, and so the streamlines are just the characteristic lines. A very interesting particular case of PDE is obtained choosing the smart intrinsic coordinate $\xi$ along flow’s vortex lines. More, one can find another 2-D velocity “quasi-potential” $\Phi_{ij}(\xi, \eta)$ satisfying the wave (vibrating string) PDE along other virtual “ij” isentropic and polytropic space curves (that is the intersection lines of the “i” isentropic surfaces with the “j” polytropic ones), that means a hyperbolic PDE (its canonical form: $\partial^2 \Phi_{ij}(\lambda, \mu, v_{0i})/\partial \lambda \partial \mu + ...$ (lower order terms) = 0), irrespective of the pseudo-flow character. The analysis for a steady flow in sections 6 and 7 is independent on $a$. Along the intersection lines of the particular isentropic surfaces ($(V, \Omega)$ sheets with “j” circular cones; $dS_1 = dS_2 = 0$, and so $dS = 0$ only), the steady rotational flow of a viscous Newtonian compressible fluid is governed by Steichen’s PDE of a 2-D velocity quasi-potential also $\Phi_{ij}(\xi, \eta, \zeta_0) = \Phi_{ij}(\lambda, \mu, v_{0i})$ – a line potential; $\zeta = \zeta_0$ is a $(V, \Omega)$ sheet, admitting this $\Phi_{ij}$.

**Remark 13.1.** The difference between a potential function $\Phi$ and a 2-D “quasi-potential” $\Phi_1$ one consists in that the first function is valid everywhere (at any point in space), whereas the second one is valid over a family of some “i” surfaces $\zeta = \zeta_0i$ only. While $V = \nabla \Phi$ is a gradient, $V_1 = \nabla \Phi_1$ is not. For the first one: $\Omega = V \times V = \nabla \times \nabla \Phi = 0$, whereas for the 2nd one: $\Omega_1 = V \times V_1 = \nabla \times \nabla \Phi_1 = k_1 \Omega_{ij} + k_1 \Omega_{ni} \neq 0$, being contained in the plane tangent to the “i” surface $\zeta = \zeta_0i$, only its component directed along the normal to this surface becoming zero: $\Omega_{\zeta_0} = (1/h_2 \cdot \eta_1 \cdot (\partial^2 \Phi_{ij}/\partial \eta \partial \eta - \partial^2 \Phi_{ij}/\partial \eta \partial \zeta) = 0$.

**Remark 13.2.** Sections 6, 7, 10 and 11 are dedicated to the analysis of some interesting particular forms of the 2-D velocity quasi-potential PDE of a certain rotational “flow”, forms taken along some space lines different from the streamlines, but having a physical & mathematical property kept along them, so that a more appropriate term would be “pseudo-flow”.

**Remark 13.3.** If a PDE has coefficients that are not constant, it is possible that it will not belong to any of the categories: elliptic, hyperbolic, parabolic, but rather be of mixed type. Besides Steichen’s PDE, a simple but important example is the Euler–Tricomi equation, used in the investigation of transonic flow: $\partial^2 u/\partial x^2 = \lambda \partial^2 u/\partial y^2$, which is called elliptic-hyperbolic because it is elliptic in region $x < 0$, hyperbolic in region $x > 0$, and degenerate parabolic on the line $x = 0$.

**Remark 13.4.** All results here obtained substantiate the existence of some 2-D quasi-potentials (velocity’s $\Phi$ and magnetic $\Xi$) even for the rotational flow, and more, for the viscous fluid flow. Not all
isentropic surfaces allow introducing such a quasi-potential, but the envelope sheets of the local planes \((\mathbf{V}, \mathbf{\Omega})\) for \(\Phi\), and \((\mathbf{H}, \mathbf{j})\) for \(\Xi\) (using the low-frequency Ampé’s law \(\mathbf{j}/k = \nabla \times \mathbf{H}\), neglecting the displacement current) only. So we can write: \(\mathbf{V} = \nabla \Phi\) and \(\mathbf{H} = \nabla \Xi\). Another 2-D quasi-potential (vortical \(X\) – the Greek “Chi”) was introduced \((\mathbf{\Omega})\) over the \((\mathbf{\Omega}, \mathbf{\Theta})\) surfaces \((\mathbf{\Theta} = \nabla \times \text{Omega})\), to study the vorticity equation for a viscous incompressible fluid ([5], [7]). All quasi-potentials \(\Phi\) satisfy Steichen’s PDE. The validity domains for these PDEs – the “ij” space lines (intersection lines of two particular isentropic sheet families – see section 11, and along Selescu’s magnetohydrodynamic vector lines: \(dR|\mathbf{S} = (\mathbf{V}, \mathbf{\Omega}) \cap (\mathbf{H}, \mathbf{j}) = (\mathbf{\Omega} \times \mathbf{V}) \times (\mathbf{j} \times \mathbf{H})\) in the inviscid MHD (see, e.g.: [2], [41], [44]) and the “i” surfaces – were theoretically established (their existence was firmly predicted). Finding them is a very difficult job and is not the subject matter of this work. The solid and solidifiable (like in the case of confluent flows) boundaries of the flow domain are always \((\mathbf{V}, \mathbf{\Omega})\) surfaces. There also is a 2-D “quasi-stream function” \(\Psi\) defined in section 2 of [49], satisfying a similar PDE as the 2-D “quasi-potential” \(\Phi\) does, the interdependence \(\Phi_i \leftrightarrow \Psi_{ci}\) being given in section 3 of [49], on the \((\mathbf{V}, \mathbf{\Omega}_i)\) sheets: \(V_{\xi_i} = \partial \Phi_i / \partial \lambda = (\partial \Psi_{ci} / \partial \mu) / \rho\); \(V_{\eta_i} = \partial \Phi_i / \partial \mu = - (\partial \Psi_{ci} / \partial \lambda) / \rho\).

So, their gradient vector lines are orthogonal each other:

\[
\nabla \Phi_i \cdot \nabla \Psi_{ci} = (\partial \Phi_i / \partial \lambda)(\partial \Psi_{ci} / \partial \lambda) + (\partial \Phi_i / \partial \mu)(\partial \Psi_{ci} / \partial \mu) = 0.
\]

Finally we can cite a remarkable sentence from [50]: “There is nothing more practical than a good theory”.

14 Notes

This paper (the last in a series dedicated to the intrinsic analytic study of the basic equations in compressible fluid mechanics) is fully original, however having as starting point another one with almost the same title (see [51]). It addresses to researchers in higher mathematics (field theory, potential theory, PDEs) and compressible fluid dynamics, MHD included. All authors use further Steichen’s PDE of the velocity potential for true potential flows only.

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The author is very grateful to Milton Van Dyke for his often-cited book (album) [20], which presents an impressive collection of about 400 selected black-and-white photographs of flow visualization in experiments, received – on his request – from researchers all over the world, these photos inspiring many interesting theoretical works.

On 16 December 2010 the Romanian Academy (proposal made by the academician Ioan M. Anton, the only specialist in Fluid mechanics and MHD in the Section of Technical Sciences of the Academy) awarded the “Traian Vuia” scientific prize for the year 2008 to this author, for a group of seven papers (entitled “Intrinsic analytic study on the isoenergetic flow of a compressible fluid”); also see the hyperlinks:
http://www.acad.ro/com2010/pag_com10_1216.htm;
http://www.incas.ro/index.php?option=com_content&view=article&id=189&Itemid=75; and

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A Simpler Velocity Quasi-Potential PDE in Fluid Mechanics, MHD included


