On optimal thermal control of an idealized room including a max-control cost term

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Abstract. We study an open-loop optimal control problem for energy efficient cooling of a simple room. The goal is to provide insights for the development of an implementable control scheme based on a Model Predictive Control strategy. The cost function includes a non-standard Mayer-term $F(\max u(t))$, that is, a term depending on the peak value of the control. Necessary conditions for an optimal control problem are formulated and it is observed that candidates of optimality can be obtained as solutions of a related two-point boundary value problem (TPBVP). Due to numerical issues a finite dimensional approximation of the optimal control problem is considered to provide approximations to the solutions of the TPBVP. The procedure is then illustrated on a numerical example.

1 Introduction

Buildings account for about 40 % of the US energy consumption, so that even modest savings are significant. The Energy Efficient Buildings Hub is a DOE sponsored effort that focuses on design and implementation of energy efficient retrofit concepts for commercial buildings in the Greater Philadelphia area [1]. Part of this effort includes detailed modeling of the interior airflow including CFD simulations [2, 3, 4]. In the present discussion we opt for a simple thermal model for a room and concentrate on an optimal control formulation [5, 6]. Our objective is to provide a benchmark for comparing implementable closed-loop control schemes based on Model Predictive Control [7].

2 Room Model

2.1 System Dynamics

To focus ideas we consider a (summer) cooling scenario with the following elements:

Mathematics Subject Classification: 49K25

Keywords: Energy Efficient Buildings
• exterior wall - exposed to the outside air,
• thermal storage - high thermal capacitance features of the building interior,
• room air - temperature in the occupied zone.

A notional view of the room is shown in Figure 1. Thermal energy storage is modeled in the solid circles which depict the
• wall interior temperature ($T_i$),
• storage temperature ($T_s$), and
• room-air temperature ($T_a$) in an occupied zone.

Four energy exchange mechanisms are modeled:

1. $q_{\text{cond}}$ - conduction through the exterior wall from $T_i$ to $T_e$
2. $q_{\text{rad}}$ - radiation from $T_i$ to $T_s$
3. $q_{\text{conv}, i}$ - convection from $T_i$ to $T_a$
4. $q_{\text{conv}, s}$ - convection from $T_s$ to $T_a$

With these energy-exchange mechanisms, the lumped dynamic model is

$$C_i \frac{dT_i}{dt} = -(q_{\text{cond}} + q_{\text{rad}} + q_{\text{conv}, i})$$
$$C_s \frac{dT_s}{dt} = (q_{\text{rad}} - q_{\text{conv}, s})$$
$$C_a \frac{dT_a}{dt} = (q_{\text{conv}, i} + q_{\text{conv}, s} - C(u)),$$

where the $C_k$ values ($k \in \{i, s, a\}$) are thermal capacitances, and $C(u)$ is the cooling delivered to the room-air using electrical input power, $u$.

![Fig. 1 Notional room configuration](image)

The conduction term is given by a simple one-dimensional model as

$$q_{\text{cond}} = \frac{kA}{\Delta} (T_i - T_e).$$

Radiant exchange between the two interior surfaces is idealized as planar and closely spaced.
\[ q_{\text{rad}} = \sigma A \left( \frac{\varepsilon_i \varepsilon_s}{\varepsilon_i + \varepsilon_s - \varepsilon_i \varepsilon_s} \right) (T_i^4 - T_s^4). \]

If the surface temperatures are reasonably close, the radiant exchange can be approximated by the linear model
\[ q_{\text{rad}} = \sigma A \hat{T}^3 \left( \frac{\varepsilon_i \varepsilon_s}{\varepsilon_i + \varepsilon_s - \varepsilon_i \varepsilon_s} \right) (T_i - T_s), \]
where \( \hat{T} \) is a suitable mean temperature. Finally, the convection terms are given by
\[ q_{\text{conv}} = (hA)_{\kappa} (T_k - T_a), \quad k \in \{i, s\}. \]

In these equations \( A, \Delta, k, h, \varepsilon_k \) are the wall surface area, thickness, conductivity, film coefficient, and emissivity, respectively; and \( \sigma \) is the Stefan-Boltzman constant.

In summary, with the linearized model for radiant exchange the dynamics of our system can be written as
\[
\dot{z}(t) = f(t, z(t), u(t)) = Az(t) + \begin{bmatrix}
\frac{kA}{C} T_i(t) \\
0 \\
\frac{-C(u(t))}{C}
\end{bmatrix},
\]
\[ u(t) \in \Omega(U) \equiv \{ \{0\} \cup [U, \bar{U}] \}, \quad (2.1) \]
where
\[ z(t) = \begin{bmatrix} T_i(t) \\ T_s(t) \\ T_a(t) \end{bmatrix}. \quad (2.2) \]

The specific structure of the \( A \)-matrix may be deduced from the previous discussion.

### 2.2 Quadratic \( \text{cop}(u) \)

The cooling function \( C(u) \) is represented as the product of the applied power \( (u) \), and a function that characterizes the system’s coefficient of performance \( (\text{cop}(u)) \). We consider a case wherein
\[ \text{cop}(u) = \alpha + \beta u + \gamma u^2 \]
with data specifications:
- \( \text{cop}(0) = \alpha > 0 \),
- \( \arg\max \text{cop}(u) = \ell, \quad U < \ell < \bar{U} \),
- \( \max \text{cop}(u) = v > \alpha \),

From these specifications we find that:
\[ \beta = \frac{2(v - \alpha)}{\ell} > 0 \quad \text{and} \quad \gamma = -\frac{(v - \alpha)}{\ell^2} < 0. \]

It’s useful to introduce a scaled control variable
\[ \hat{u} \equiv \frac{u}{\ell}, \]
so that
\[ \text{cop}(\hat{u}) = \alpha + (v - \alpha)\hat{u} [2 - \hat{u}] , \]

and

\[ C(\hat{u}) = \ell \hat{u} \text{cop}(\hat{u}) . \]  

(2.3)

In the next section we formulate an optimal control problem for these dynamics. In the following we drop the \( \hat{u} \) notation so that \( u \) is the scaled applied power.

3 Optimal Control Problem

Whereas the primary focus of our optimization study is minimal energy use, to avoid trivialities it is necessary to place some restrictions on the temperature histories. To this end we formulate a discomfort metric, namely

\[ D = \int_0^{t_f} c(t) \Psi(T_a(t), T_{\text{min}}, T_{\text{max}}) \, dt , \]

where

\[ \Psi(T, T_{\text{min}}, T_{\text{max}}) = \begin{cases} (T - T_{\text{min}})^2, & T < T_{\text{min}} \\ (T - T_{\text{max}})^2, & T > T_{\text{max}} \\ 0, & \text{otherwise} \end{cases} , \]

and where \( c \) is a zero-one function, the characteristic function of the occupied time interval. With \( T_{\text{min}} = T_{\text{max}} \), the function \( \Psi \) is a quadratic that penalizes variation in room-air temperature, and with \( T_{\text{min}} < T_{\text{max}} \) there is a finite comfort range with zero quadratic penalty.

In formulating the power cost we admit dependence on peak-demand and time-varying power rates to define our cost-functional as

\[ J[u] = F(U) + \ell \int_0^{t_f} r(t) u(t) \, dt + \mu D D , \]

(3.1)

where \( r > 0 \) is a given function specifying the time-varying cost of power. The parameter \( \mu > 0 \) permits a trade-off of the power-cost and discomfort metrics. The function \( F \) admits dependence on the maximum value; we suppose that \( F \) is convex on its domain. To study this feature we include \( U \) as a 4th state and incorporate a state-dependent control constraint

\[ \dot{U} = 0 , \quad u(t) - U / \ell \leq 0 . \]

We anticipate that the exterior wall temperature \( T_e \), the power-rate \( r \), and the characteristic function \( c \) will be given over a 24 hour period and that the parameter \( \mu_D > 0 \) is given. We take \( t_f = 24\text{h} \) and seek periodic boundary conditions \( z(0) = z(t_f) = z_0 \) and a control function \( u^* \) to minimize the cost (3.1) subject to the dynamics (2.1).

4 Necessary Conditions

We apply the Minimum Principle [5, 6] to our problem and begin by defining the variational Hamiltonian
\[ H(t, \lambda_0, \lambda_z, \lambda_{\mathcal{T}}, z, \bar{U}; u) = \left\langle \begin{pmatrix} \lambda_0 \\ \lambda_z \end{pmatrix}, \begin{pmatrix} f_0(t, z, u) \\ \Gamma(t, z, u) \end{pmatrix} \right\rangle \\
+ \nu \left( u - \frac{\bar{U}}{\ell} \right) \]
\[ = \langle \lambda_z, Az \rangle - \lambda_a \frac{C(u)}{C_a} \\
+ \lambda_0 \left( r(t) \frac{\ell}{u} + \mu_{DC}(t) \Psi(T_a) \right) \\
+ \nu \left( u - \frac{\bar{U}}{\ell} \right) \quad \text{(4.1)} \]

where \( z \) is given by (2.2) and the components of the adjoint variable \( \lambda_z \) are defined similarly. The Minimum Principle requires that the scalar \( \lambda_0 \geq 0 \); we assume that \( \lambda_0 \neq 0 \) and normalize the adjoint system with the choice \( \lambda_0 = 1 \). Note that in [5, 6] the authors require \( \lambda_0 \leq 0 \) and develop a Maximum Principle. The last term in (4.1) accounts for the state-dependent control constraint; \( \nu \) is the Valentine multiplier [8].

### 4.1 Adjoint System

Evolution of the co-state \( (\lambda_z) \) is governed by the inhomogeneous linear system:

\[ \dot{\lambda}_z(t) = -A^T \lambda_z(t) - \begin{bmatrix} 0 \\ 0 \\ \mu_{DC}(t) \frac{\partial \Psi}{\partial T_a} \end{bmatrix} \quad \text{(4.2)} \]

For the augmented state \( (\bar{U}) \) the corresponding adjoint equation is

\[ \dot{\lambda}_{\mathcal{T}} = \nu / \ell \quad \text{(4.3)} \]

On time-subarcs wherein \( u^*(t) = \frac{\bar{U}}{\ell} \), \( \nu \) follows from \( \frac{\partial H}{\partial u} = 0 \), namely

\[ \nu = \frac{\lambda_a(t)}{C_a} C'(u) - \ell r(t) \quad \text{(4.4)} \]

otherwise, \( \nu = 0 \).

### 4.2 Optimality

We consider that part of the variational Hamiltonian that depends (explicitly) on the control \( (u) \). Noting that the electrical power cost function \( (r(t)) \), and the coefficient of performance parameter \( (\ell) \) are positive, we factor these out

\[ \hat{H}(u) \equiv \left( \frac{1}{r(t) \ell} \right) H_{\text{cont}}(u) = u - \left( \frac{\lambda_{\mathcal{T}}(t)}{C_a r(t)} \right) \frac{C(u)}{\ell} = \hat{\lambda}_a(t) \quad \text{(4.5)} \]

where \( C(u) \) is the cooling rate from Equation (2.3) (recall that \( \nu \left( u - \frac{\bar{U}}{\ell} \right) = 0 \) by complementarity).

The Minimum Principle requires that we characterize the value(s) of the control that minimize \( \hat{H} \) over the set \( \Omega(\bar{U}) \). Since \( \hat{H} \) is continuous and the domain \( \Omega(\bar{U}) \) is compact, minimizers exist; since \( \hat{H} \) is smooth there are four possibilities to consider for extremal control \( (u^*) \):
1. \( u^* = U \), which requires that \( \frac{\partial \hat{H}}{\partial u} |_U \geq 0 \);
2. \( u^* = \overline{U} \), which requires that \( \frac{\partial \hat{H}}{\partial u} |_{\overline{U}} \leq 0 \);
3. \( u^* = u_{int} \) occurs at an interior point with \( \frac{\partial \hat{H}}{\partial u} |_{u_{int}} = 0 \) and \( \frac{\partial^2 \hat{H}}{\partial u^2} |_{u_{int}} \geq 0 \);
4. \( u^* = 0 \), that is, the isolated point in \( \Omega(\overline{U}) \) - zero power.

For current purposes it’s useful to exploit the general structure of the variational Hamiltonian (4.1) and to interpret the \( \arg \min H \) operation geometrically. The control \( (u) \) dependent part of the variational Hamiltonian (4.1) is

\[
\hat{H}(u) = \left( \frac{1}{\hat{\lambda}_a} \right) \cdot \left( \frac{u}{C(u)} \right).
\]

The vector on the right in this inner-product is the (augmented) state-rate and captures the time-rates of the control-dependent parts of the cost function and the state (normalized as in 4.5). The locus of admissible points is shown in Figure 2 for a case with \( \overline{U}/\ell = 1.4 \). This locus of admissible state-rates is called the Velocity Set. In the present case it consists of the point at the origin, and an operating line of points corresponding to control values \( u \in [\overline{U}/\ell, \overline{U}/\ell] \). Note that the other terms in the variational-Hamiltonian are independent of the control and do not affect the \( \min -H \) operation.

Next, consider a fixed vector \( \hat{\lambda}_a \equiv \left( \frac{1}{\hat{\lambda}_a} \right) \), where \( \hat{\lambda}_a \equiv \lambda_a(t) / C_a(r(t)) \) as in Equ’n (4.5). The orthogonal complement \( \{ \hat{\lambda}_a \}^\bot \) is the subspace of vectors orthogonal to (the span of) \( \hat{\lambda}_a \), and for points in this subspace we have \( H_c = 0 \) (see Figure 3). Any translation of this subspace along the \( + \hat{\lambda}_a \) direction has \( H_c = \chi \), a positive real, whereas any translation along the opposite direction is a set of points with negative real values for the function \( H_c \).

The ideas underlying Figures 2 and 3 are combined in Figure 4. In the case shown the minimizing control is \( u^* = 1.2 \), which is between the control for maximum cop and the upper bound \( \overline{U}/\ell = 1.4 \). Figure 5 shows the case \( \hat{\lambda}_a = \hat{\lambda}_a^II \) wherein the separating plane is tangent to the Velocity Set at \( u = 1.4 = \overline{U}/\ell \). For larger values of \( \hat{\lambda}_a \) the separating plane still goes through the operating point at
$u = 1.4$, although the plane is no longer tangent to the operating line. As $\tilde{\lambda}_a$ decreases from $\tilde{\lambda}_a^II$ the vector $\hat{\lambda}_a$ rotates clockwise, the separating plane remains tangent to the Velocity Set and the operating point decreases from $U/\ell$. This continues until we reach the case $\tilde{\lambda}_a = \tilde{\lambda}_a^I$ shown in Figure 6. Note that in this case the separating plane also passes through the operating point at the origin. Further decreasing $\tilde{\lambda}_a$ (i.e. further clockwise rotation of $\hat{\lambda}_a$) implies optimal operation at $u^* = 0$.

From this analysis we find there are two critical values of $\tilde{\lambda}_a$

$$\tilde{\lambda}_a^I = \frac{1}{\frac{\partial (C(u)/\ell)}{du}} |_{\ell} = \frac{1}{v},$$

and

$$\tilde{\lambda}_a^II = \frac{1}{\frac{\partial (C(u)/\ell)}{du}} |_{\ell/\ell} = \frac{1}{\alpha + 4(v - \alpha)U/\ell - 3(v - \alpha) (U/\ell)^2}.$$
We assume that $\frac{U}{\ell} > 1$ so that $\tilde{\lambda}^\text{II}_a > \tilde{\lambda}^\text{I}_a$. If the upper bound is less than the value for maximum coefficient of performance ($\bar{\lambda} < \ell$) then extremal controls take values at either $\frac{U}{\ell}$ or at 0.

In summary the results of the min $H$-operation are

$$u^*(\tilde{\lambda}_a) = \begin{cases} 0 & \text{if } \tilde{\lambda}_a < \tilde{\lambda}^\text{I}_a \\ u_{\text{int}} & \text{if } \tilde{\lambda}^\text{I}_a < \tilde{\lambda}_a < \tilde{\lambda}^\text{II}_a \\ \frac{U}{\ell} & \text{if } \tilde{\lambda}^\text{II}_a \leq \tilde{\lambda}_a \end{cases}$$

(4.6)

where

$$u_{\text{int}} = \frac{2 + \sqrt{4 - 3p}}{3} \quad \text{and} \quad p \equiv \frac{1}{\lambda_0} - \alpha \left(\frac{v}{v - \alpha}\right).$$
The choice of sign (+) guarantees that \( \frac{\partial^2 H_c}{\partial u^2} > 0 \), so that \( u_{\text{int}} \) is (at least) a local minimizer. Note that whereas it can be shown that \( u^* \) is a continuous function of \( \tilde{\lambda}_a \), on the interval \( \left[ \tilde{\lambda}_a^1, \infty \right) \) the definition of \( \tilde{\lambda}_a \) involves the power-weighting factor \( (r) \) which need not be a continuous function of time. The interior control (scaled) takes values in the interval \( \left[ 1, \frac{U}{\ell} \right] \). Thus, it is never optimal to take control values in the interval \( \left[ \frac{U}{\ell}, 1 \right) \). This statement is equivalent to the classical Weierstrass Necessary Condition [9, see Appendix C.4].

In constructing Figures 2 - 6 we have used \( \alpha = 4 \), \( v = 8 \). Nonetheless, the geometric features, such as the origin lying on the tangent line in Figure 6, and the characterization of extremal controls in Equation (4.6) are generic.

### 4.3 Transversality

For our periodic boundary condition on the \( z \)-state, the transversality conditions are [10]:

\[
\lambda_z(t_f) = \lambda_z(0) . \tag{4.7}
\]

The \( 4\text{th} \)-state \( (U) \) is unspecified at \( t = 0 \), so that

\[
\lambda_{U}(0) = 0 ,
\]

whereas at \( t = t_f \) we have

\[
\lambda_{U}(t_f) = \frac{\lambda_0}{F'(U)} .
\]

Combining these conditions with (4.3, 4.4) leads to

\[
F'(U) = \int_0^{t_f} \lambda_{U}(t) \, dt = \int_{t_u^* = U} \left[ \lambda_a(t) \frac{C(U_a)}{C_a} - r(t) \right] \, dt . \tag{4.8}
\]

Equation (4.8) is an optimality condition for the unknown parameter \( U \).

### 5 Computational Approaches

#### 5.1 Indirect Method: Optimality System

The dynamic model (2.1), the adjoint system (4.2), the optimality result (4.6), along with the periodic boundary condition and associated transversality condition (4.7) form a two-point boundary value problem (TPBVP) in the composite pair \( (z, \lambda_z) \). Since the Minimum Principle is a necessary conditions, solutions of this TPBVP are candidates for optimality. Unfortunately, numerical solution of this TPBVP can be difficult to obtain. One reason for this behavior is the Hamiltonian structure of the combined state/adjoint system. Specifically, if \( \sigma \in C \) is an eigenvalue of the (linearized) system, then so is \( -\sigma \). Indeed, for typical system parameters we find the eigenvalues of the state/adjoint system are:

\[
\sigma = \pm 0.015/h, \pm 0.153/h, \pm 19.579/h .
\]

For the fastest growing of these over the time interval \( [0, 24h] \) the growth is
While a parallel-shooting approach may mitigate the computational sensitivity, it seems that any approach based on solving the boundary-value problem ab initio is likely to be a challenge. For this reason it’s useful to consider alternative approaches to develop a plausible initial estimate for the TPBVP.

5.2 Direct Method: NLP Formulation

In a direct method one constructs a finite-dimensional approximation to the underlying optimal control problem. In early implementations of this approach [11] the control history was represented by a finite-dimensional parameterization; the parameters are the unknowns in the NLP problem. Evaluation of cost/constraint functions required numerical solution of an initial value problem arising from the state dynamics. In later formulations both the state and control histories are parameterized and the dynamics are enforced approximately by a collocation method. The collocation defects are driven to zero as equality constraints in the NLP formulation. A consistent formulation leads to a Nonlinear Programming Problem (NLP) wherein the resulting solution yields credible estimates for the states and the adjoints arising in the TPBVP of the indirect method [14].

In the present case we use the second approach and discretize the dynamic model (2.1) using the implicit midpoint rule (2\textsuperscript{nd}-order). The NLP variables are the values of the (3) states and (1) control at N grid points uniformly spaced on the interval [0, 24 h] and the cost function (3.1) is approximated by numerical quadrature. The MATLAB procedure fmincon was used with the active set algorithm. In this formulation the 3*N collocation conditions are nonlinear equality constraints in the NLP.

Shown in Figure 7 are the resulting temperature (state) histories. Note that the zone-air temperature \( T_a \), in red) initially decreases from midnight to about 04:00 h when the cooling goes off. Cooling resumes at about 07:00 and remains on until about 09:00. The zone-air temperature then increases until it hits the upper bound \( T_{max} = 25^\circ C \) at about 13:30. From that point until the end of the occupied time the cooling cycles. At about 17:30 the cooling goes off and the zone-air temperature increases so that it attains the upper bound just at the end of the occupied period (18:30 h). Finally, cooling comes on at about 21:30 and continues to cool until about 04:00 the next morning. The corresponding (cooling-power) control is displayed in Figure 8. The horizontal dashed (red) line in this figure is the value \( u = \ell \), the power setting for maximum coefficient of performance. Note that whereas most of the time this NLP solution observes our prescription that optimal controls never take values in the range \([U, \ell]\), it does violate this condition around 08:00 and again around 16:00. It’s possible that our ‘solution’ is only local minimizer for the NLP problem. This observation reinforces the need for conditions that test optimality for the optimal control problem.

6 Conclusions

In our model with quadratic \( \text{cop}(u) \) admissible controls can take values in \( \Omega(U) = \{0 \cup [U, U]\} \), where \( 0 < U < \ell < U \) and \( \ell = \arg\max U \text{cop}(u) \). Analysis shows that optimal controls never take values in \([U, \ell]\); that is it is never optimal to use positive power less than that required for maximum cop. Preliminary numerical evidence indicates that optimal cooling includes pre-cooling, wherein the zone-air temperature is lowered to near its minimum-allowed value prior to occupancy. Cooling then remains off until the zone-air temperature reaches its maximum-allowed value. A sequence of
cooling-on/cooling-off segments continues to maintain the zone-air temperature near its maximum-allowed value until with the cooling off the zone-air temperature reaches the maximum-allowed value at the end of the occupied period. After a cooling-off period the pre-cooling cycle begins again.

Finally, note equation (4.8), our optimality condition for the unknown parameter $\overline{U}$ in the case with a max-power Mayer cost term. If an optimal $\overline{U}$ is known for given parameter values, then increasing the power-cost ($r(\cdot)$) will decrease the critical value of $F'$, and (assuming $F$ is convex) leads to a lower optimal peak-value. This argument is incomplete since the $t -$ set over which integration occurs depends on $\overline{U}$.
Acknowledgements

Supported in part by DOE under Award Number DE-EE0004261 and in part by AFOSR under Grant FA9550-10-1-0201.

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