Second sound propagation in laminar and turbulent superfluid $^4$He via extended thermodynamics

Lucia Ardizzone

Dipartimento di Ingegneria Elettrica, Elettronica e delle Telecomunicazioni, di Tecnologie Chimiche, Automatica e Modelli Matematici (DIEETCAM)
Università degli Studi di Palermo, Viale delle Scienze, 90128 Palermo, Italy

Corresponding Author: lucia.ardizzone@unipa.it

Abstract. In this work, previous results on the complex propagation of second sound in laminar and turbulent superfluid $^4$He are reported. Furthermore the case of inhomogeneous superfluid turbulence is examined and, in particular, the wave propagation is investigated in the presence of an anisotropic vortex tangle. Two cases of physical interest are considered: wave front collinear and orthogonal to the heat flux.

1 Introduction

It is well known that the lambda point ($T_\lambda \simeq 2.17$ K, at vapor pressure) is the temperature below which normal fluid helium (helium I) undergoes a transition to superfluid helium II. The properties of helium I are those of an ordinary fluid, instead in the behavior of helium II one observes a variety of singular phenomena that have no counterpart in a normal fluid. An example of non-classical behavior is heat transfer in counterflow experiments, characterized by no net matter flow, but only heat transport. If helium II is used, and the heat flux inside the channel is not too high, the temperature gradient is so small that it cannot be measured, so indicating that the liquid has an extremely high thermal conductivity (several hundred times higher than copper). This effect explains the remarkable ability of helium II to remove heat and makes it important in engineering applications: in fact, liquid helium is often used in the aerospace industry as refrigerant. For example, in the European space telescope Herschel it is necessary to reach temperatures of the order or less than 1 K. In addition, the liquid helium is used for the refrigeration of large magnets, such as those found at CERN (European Organization for Nuclear Research). Studies on magnetic levitation trains (maglev) obtained for electrodynamic suspension (EDS) that uses refrigeration techniques to liquid helium are in progress.

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Recently, several space agencies, including NASA (National Aeronautics and Space Administration) and ESA (European Space Agency), are engaged in research on maglev to develop an economical method of space launcher, allowing a considerable saving of fuel.

A first important issue to be addressed in the problems of cryogenic refrigeration is the onset of turbulence. Indeed, the presence of heat flow can produce the formation of microscopic quantized vortices and modify the thermal conductivity of the superfluid [1, 2, 3, 4]. These problems have already been addressed in [5, 6, 7, 8, 9], where it has been proposed a mathematical model for turbulent helium II in presence of heat flow.

Another example of non-classical behavior is the possibility to propagate the second sound. Second sound is a quantum mechanical phenomenon in which heat transfer occurs by wave-like motion, rather than by the more usual mechanism of diffusion. It is known as second sound because the wave motion of heat is similar to the propagation of sound in air.

Our aim, in this paper, is to examine the propagation of second sound in superfluid $^4$He. We will make use of a non-standard one-fluid model of superfluids deduced by Extended Thermodynamics. In Section 2, the propagation of heat waves is examined both in the laminar and in turbulent regimes; in Section 3 the propagation of second sound in inhomogeneous turbulence is examined, also in anisotropic turbulent flows.

## 2 One-fluid model of liquid helium II

Extended Thermodynamics (ET) is a macroscopic theory of non-equilibrium processes, which has been formulated in various ways in the last decades [10, 11, 12, 13]. The main difference between the ordinary thermodynamics and the ET is that the latter uses dissipative fluxes, besides the traditional variables, as independent fields. As a consequence, the assumption of local equilibrium is abandoned in such theories. In the study of non equilibrium thermodynamic processes, an extended approach is required when one is interested in sufficiently rapid phenomena, or else when the relaxation times of the fluxes are long; in such cases, a constitutive description of these fluxes in terms of the traditional field variables is impossible, so that they must be treated as independent fields of the thermodynamic process.

### 2.1 Laminar flow

In previous papers, the ET was applied to formulate a non-standard one-fluid model of liquid helium II, for laminar flows [14, 15]. This model chooses the mass density $\rho$, the velocity $\mathbf{v} = (v_i)$, the absolute temperature $T$ and the heat flux density $\mathbf{q} = (q_i)$ as fundamental fields that depend on $\mathbf{x} = (x_i)$ and $t$. For sake of simplicity, because in this paper our aim is to study the second sound propagation in superfluids, we will suppose the fluid homogeneous and at rest and we will neglect the small coupling between the temperature and the pressure wave, due to the small thermal expansion. Therefore in order to describe the propagation of the second sound it is sufficient to consider only the evolution equations for the absolute temperature $T$ and the heat flux density $\mathbf{q}$. Neglecting dissipative phenomena, the equations for these two fields read [14, 15]:

$$
\begin{align*}
\dot{T} + \frac{1}{\rho c_v} \frac{\partial q_i}{\partial x_i} &= 0, \\
\dot{q}_i + \zeta \frac{\partial T}{\partial c_v} &= 0,
\end{align*}
$$

(2.1)
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where $c_V$ is the specific heat at constant volume, and $\zeta^* = \lambda_1/\tau_1$ the ratio between the thermal conductivity $\lambda_1$ and the relaxation time of the heat flux $\tau_1$; as it was shown in [15], in helium II both $\lambda_1$ and $\tau_1$ are extremely high, but their ratio is finite and determines the second sound velocity. The dot denotes the material time derivative, summation convention on repeated indices will be used throughout this work (i.e. $\frac{\partial q}{\partial t} = \nabla \cdot \mathbf{q}$).

The model describes the propagation of a temperature wave (the second sound), whose velocity is as follows [14, 15]:

$$V^*_2 = \sqrt{\frac{\zeta^*}{\rho c_V}}.$$  (2.2)

However, experiments show that dissipative effects are present in the propagation of second sound in liquid helium II, also in the absence of vortices. In order to take into account these effects, a thermal stress tensor $\mathbf{P}_V$ was introduced in [15], whose expression was found to be:

$$[\mathbf{P}_V]_{ik} = 2\beta T\lambda_2 \frac{\partial q_{<i}}{\partial x_k} + \beta' T\lambda_0 \frac{\partial q_j}{\partial x_j} \delta_{ik}$$  (2.3)

In this equation $\lambda_0$ and $\lambda_2$ are the bulk and the shear viscosity, while $\beta$ and $\beta'$ are coefficients appearing in the general expression of the entropy flux in ET which take into account the dissipation of thermal origin. Angular brackets denote the tensor deviator, i.e. the symmetric traceless part with respect to the indices included in term (i.e. $\frac{\partial q_{<i}}{\partial x_k} = \frac{1}{2} \left( \frac{\partial q_i}{\partial x_k} + \frac{\partial q_k}{\partial x_i} \right) - \frac{1}{3} \frac{\partial q_j}{\partial x_j} \delta_{ik}$).

Introducing the tensor (2.3), the field equations (2.1) are modified as follow:

$$\begin{cases} 
\frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q}{\partial t} = 0, \\
\frac{\partial q_i}{\partial t} + \zeta^* \frac{\partial T}{\partial x_i} - \lambda_0 \beta T^3 T^3 \zeta^* \frac{\partial q}{\partial x_i} [\frac{\partial q_j}{\partial x_j}] + 2\lambda_2 \beta T^3 T^3 \zeta^* \frac{\partial q}{\partial x_j} [\frac{\partial q_{<i}}{\partial x_k}] = 0.
\end{cases}$$  (2.4)

The propagation of small amplitude waves was studied in [15]. Under the hypothesis of small losses approximation, one sees that the velocity of the second sound is identical to that found in the absence of dissipation, while the attenuation coefficient is:

$$k^{(2)}_s = \frac{\omega^2 T^3 T^3 \zeta^*}{2 (V^*_2)^3} \left( \lambda_0 \beta^2 + \frac{4}{3} \lambda_2 \beta^2 \right)$$  (2.5)

where $\omega$ is the frequency.

2.2 Turbulent flow

One of the most typical effects in liquid helium II is the so-called counterflow superfluid turbulence [1, 2, 3, 4]. This new type of turbulence, known as quantum turbulence, is described as a chaotic tangle of quantized vortices of equal circulation $\kappa$; $\kappa$ is called quantum of vorticity and results $\kappa = \hbar/m_4$, where $\hbar$ is the Planck constant, and $m_4$ the mass of $^4$He atom: $\kappa \simeq 9.97 \times 10^{-3} \text{cm}^2/\text{s}$.

In the present subsection, our attention is focused on the study of the action of vortices on the propagation of second sound. To this aim the one-fluid model of liquid helium II derived in the framework of ET must be modified in order to take into account the presence of vortices.

A first thermodynamic study of these interesting phenomena was made in [5], where the attention was restricted to stationary situations, in which the vortex filaments were supposed fixed. We will recall now briefly the study presented in [5].
To take into account the dissipation due to the vortices following [5], we introduce in the field equations (2.1) a new dissipative tensor $P_{\omega}$, for which a constitutive relation must be written. Furthermore, we neglect $P_V$ as compared to $P_{\omega}$, because the mutual friction effects are much greater than thermal bulk and shear forces acting inside the superfluid.

Let us now reformulate the evolution equation for the heat flux $q$. The experimental data show that the extra attenuation due to the vortices is independent on the frequency. Therefore, a rather natural generalization of the second equation in system (2.1) for the time evolution of the heat flux $q$ is:

$$\dot{q} + \zeta^* \nabla T = -P_{\omega} \cdot q.$$  

This relation is written in an inertial system; the influence of the vortices on the dynamics of the heat flux is modelled by the last term in the r.h.s. of (2.6).

To close the set of equations, we need a constitutive relation for the tensor $P_{\omega}$. To this purpose, we note that the presence of quantized vortices leads to a force of interaction with the excitations in the superfluid known as mutual friction. From a microscopic point of view, the major source of mutual friction results from the collision of rotons with the cores of vortex lines: the quasiparticles scatter off the vortex filaments and transfer momentum to them. The collision cross-section is clearly a function of the direction of the roton drift velocity relative to the vortex line: it is a maximum when the roton is travelling perpendicular to this line and a minimum (in fact zero) when the roton moves parallel to it. In counterflow situations, we have:

$$P_{\omega} = \frac{1}{2} < \omega > \left[ B < U - s' \otimes s' > \right],$$

where brackets denote macroscopic averages. The unspecified quantities introduced in (2.7) are the following: $\omega$ is the microscopic vorticity vector, $\omega = |\omega|$; $B$ is the Hall-Vinen coefficient [16]; $s'$ is a unit vector tangent to the vortices, $U$ the unit second order tensor. Finally, the quantity $< \omega >$ depends on the average vortex line length per unit volume, $L$, through the simple proportionality law:

$$< \omega > = \kappa L,$$

where $\kappa$ is the quantum of vorticity.

Neglecting the thermal bulk and shear forces and under the hypothesis of small thermal dilatation (which in helium II are indeed very small), the linearized system of field equations for liquid helium II, in an inertial frame, in absence of external force, is [5]:

$$\begin{cases}
\frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q_j}{\partial x_j} = 0 \\
\frac{\partial q_i}{\partial t} + \zeta^* \nabla q_i = - (P_{\omega} \cdot q)_i.
\end{cases}$$

A more general model, in which thermal bulk and shear forces acting inside the superfluid, that we have described using the thermal stress tensor $P^k$, are taken into account is:

$$\begin{cases}
\frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q_j}{\partial x_j} = 0 \\
\frac{\partial q_i}{\partial t} + \zeta^* \frac{\partial T}{\partial x_i} - \lambda_0 \beta^2 T^3 \zeta^* \frac{\partial T}{\partial x_i} + 2 \lambda_2 \beta^2 T^3 \zeta^* \frac{\partial q_j}{\partial x_i} \left[ \frac{\partial q_j}{\partial x_i} \right] = - (P_{\omega} \cdot q)_i.
\end{cases}$$

We suppose, now, that the vortex distribution is described as an isotropic tangle. This allows us to assume that the microscopic vorticity $\omega$ (hence its unit vector $s'$) is isotropically distributed, so that:
As a consequence, the pressure tensor (2.7) takes the simplified form
\[ \mathbf{P}_\omega = \frac{1}{3} \mathbf{B} <\omega> \mathbf{U}, \]
where \(<\omega>\) depends on the average vortex line length \(L\) per unit volume, through the simple proportionality law (2.8). We obtain therefore
\[ \sigma^q = -\frac{1}{3} \kappa BLq. \]

3 Inhomogeneous superfluid turbulence

As shown in the previous section, a fundamental scalar quantity, needed in the macroscopic description of superfluid turbulence is the line density \(L\), defined as the total length of vortex lines per unit volume. The dimensions of \(L\) are \((\text{length}/\text{length}^3) = (\text{length})^{-2}\). Since the vorticity is quantized, the increase of turbulence is manifested as an increase of the total length of the vortex lines. Therefore, the dynamic evolution of this quantity is a central aspect in quantum turbulence.

Here, we will study wave propagation in liquid helium II in the presence of a vortex tangles. Experiments show that in this case with good approximation only the fields \(T, q\) and \(L\) are involved. The evolution equations for these fields can be obtained by the model formulated in [6] considering the fluid at rest and neglecting the dependence of the pressure on the vortices. In this case, as one can see in [7], the temperature wave is propagating independently by the pressure wave.

For these fields the following evolution equations were obtained [6]:
\[
\begin{align*}
\rho \dot{e} + \frac{\partial e}{\partial t} &= 0, \\
\dot{q}_j + \zeta \frac{\partial T}{\partial x_j} + \chi \frac{\partial q}{\partial x_j} &= \sigma^q_j, \\
\dot{L} + \nu \frac{\partial L}{\partial q} &= \sigma^L.
\end{align*}
\]
In (3.1) \(e\) is the specific internal energy that depends on temperature \(T\) and on the line density \(L\); \(\sigma^L\) is the line density production term; \(\zeta, \chi\) and \(\nu\), are three coefficients which describe the coupling between the heat flux and the line density \(L\). Also these coefficients may depend, in principle, on \(T\) and \(L\); however, for our purpose, it is sufficient to consider them as constant quantities. In [6] (see also [8]) the physical meaning of them and their possible experimental determination was outlined.

Observe that \(E = \rho e\) is the internal energy density of the whole system, which can be considered as sum of the energy \(E_0\) of the helium background (normal and superfluid components) and of the energy of the vortex tangle \(E_V = \epsilon_V L\):
\[ \epsilon(T, L) = \epsilon_0(T) + \epsilon_V L, \]
where \(\epsilon_V\) is the energy per unit length of the vortex, also called the vortex tension; here for this quantity, we will assume that the vortex line tension is proportional to the square of the quantum of vorticity, approximation often made in the study of turbulence in superfluids [1]. We will choose therefore:
\( \epsilon_V = c p \kappa^2, \) \hspace{1cm} (3.3)

where \( c \) is a dimensionless coefficient of the order of unity.

For the heat flux production term \( \sigma^q_j \), the following constitutive relation was chosen in [6] (identical to that proposed in [5]):

\[
\sigma^q_j = - P^o_{ij} q_j, \quad \text{where} \quad P^o_{ij} = \frac{1}{2} \kappa LB < \delta_{ij} - s'_i s'_j > \hspace{1cm} (3.4)
\]

To determine the line density production term \( \sigma^L \), we recall the evolution equation for \( L \) formulated by Vinen [17], in homogeneous superfluid turbulence. Vinen assumes that the time derivative of \( L \) is composed of two opposite contributions:

\[
\frac{dL}{dt} = \left[ \frac{dL}{dt} \right]_f - \left[ \frac{dL}{dt} \right]_d, \hspace{1cm} (3.5)
\]

the first responsible for the growth of \( L \), the second for its decay. Vinen assumes that the term \( \left[ \frac{dL}{dt} \right]_f \) depends on the instantaneous value of \( L \) and the force \( f \) between the vortex line and the gas of excitations which is proportional to the modulus of the heat flux \( q \), obtaining \( \left[ \frac{dL}{dt} \right]_f = \tilde{a} |q| L^{3/2} \). The form of the term responsible for the vortex decay was determined assuming that Feynman’s model of vortex breakup is analogous to Kolmogorov’s cascade in classical turbulence, obtaining \( \left[ \frac{dL}{dt} \right]_d = \tilde{b} L^2 \). Following Vinen, in this paper we will choose for the production term in the evolution equation for the line density \( L \), the Vinen’s expression:

\[
\sigma^L = - \tilde{b} L^2 + \tilde{a} |q| L^{3/2}. \hspace{1cm} (3.6)
\]

where the coefficients \( \tilde{a} \) and \( \tilde{b} \) are positive quantities dependent on the temperature \( T \) of helium II.

### 3.1 Wave propagation in inhomogeneous isotropic turbulent superfluid \(^4\)He

In this section, we will assume isotropy in the vortex tangle, in such a way that the vorticity tensor \( P^o_{ij} \) and the production term in the heat flux equation have the simple expressions:

\[
P^o_{ij} = \frac{1}{3} \kappa LB \delta_{ij}, \quad \sigma^q_i = - \frac{1}{3} \kappa BL q_i, \hspace{1cm} (3.7)
\]

As we can easily see, a stationary solution of system (3.1) is:

\[
q = q_0 = (q_1^0, 0, 0), \quad L = L_0 = \frac{\alpha^2}{B^2} |q_1^0|^2, \quad T = T_0 - \frac{KL q_1^0}{\xi_0} x_1, \hspace{1cm} (3.8)
\]

with \( q_1^0 > 0 \) and \( K = \frac{1}{3} \kappa B \), where \( \kappa \) is the quantum of vorticity and \( B \) the Hall-Vinen coefficient.

To study the wave propagation in a neighborhood of this solution, we substitute \( \sigma^q_i \) and \( \sigma^L \) with the approximate expressions:

\[
\sigma^q_1 \approx - K \left[ L_0 q_1 + q_1^0 (L - L_0) \right],
\]
\[
\sigma^q_2 \approx - K L_0 q_2,
\]
\[
\sigma^q_3 \approx - K L_0 q_3, \hspace{1cm} (3.9)
\]
\[
\sigma^L \approx - \left[ 2 \tilde{b} L_0 - \frac{3}{2} \tilde{a} q_1^0 L_0^{1/2} \right] (L - L_0) + \tilde{a} q_1^0 L_0^{3/2} (q_1 - q_1^0),
\]
obtaining:

\[
\begin{cases}
\rho \varepsilon_T \frac{\partial T}{\partial t} + \rho \varepsilon_L \frac{\partial L}{\partial t} + \frac{\partial q}{\partial t} = 0 \\
\frac{\partial q_1}{\partial t} + \sigma \frac{\partial T}{\partial t} + \chi \frac{\partial L}{\partial t} = -K[L_0 q_1 + q_1^0 (L - L_0)] \\
\frac{\partial q_2}{\partial t} + \sigma \frac{\partial T}{\partial t} + \chi \frac{\partial L}{\partial t} = -KL_0 q_2 \\
\frac{\partial q_3}{\partial t} + \sigma \frac{\partial T}{\partial t} + \chi \frac{\partial L}{\partial t} = -KL_0 q_3 \\
\frac{\partial L}{\partial t} + \nu \frac{\partial q}{\partial t} = - \left[ \frac{3}{2} \beta L_0 \right] (L - L_0) + \tilde{\alpha} q_1^0 L_0^{1/2} (q_1 - q_1^0)
\end{cases}
\]  

(3.10)

where \( \varepsilon_T = c_V = \partial \varepsilon / \partial T, \varepsilon_L = \varepsilon_V = \partial \varepsilon / \partial L \).

Consider the propagation of harmonic plane waves, seeking for solutions of system (3.10) of the form

\[
\begin{align*}
T &= T_0 - \frac{KL_0 q_1^0}{\sigma} x_1 + \tilde{T} e^{i(k \mathbf{n} \cdot \mathbf{s} - \omega t)} \\
q_j &= q_j^0 + \tilde{q}_j e^{i(k \mathbf{n} \cdot \mathbf{s} - \omega t)} \\
L &= L_0 + \tilde{L} e^{i(k \mathbf{n} \cdot \mathbf{s} - \omega t)}
\end{align*}
\]

(3.11)

where \( q_j^0 = (q_j^0, 0, 0) \), \( k = k_x + ik_y \) is the wave-number, \( \omega = \omega_s + i \omega_t \) the frequency and \( \mathbf{n} = (n_i) \) the unit vector orthogonal to the wave front.

Inserting (3.11) in the linearized field equations (3.10), and setting:

\[
N_1 = KL_0, \quad N_2 = 2 \tilde{\beta} L_0 - \frac{3}{2} \tilde{\alpha} q_1^0 L_0^{1/2}, \quad N_3 = K q_1^0, \quad N_4 = \tilde{\alpha} q_1^0 L_0^{3/2},
\]

we obtain the following algebraic set of equations for the amplitudes:

\[
\begin{align*}
- |\rho \varepsilon_T|_0 \omega T + |\rho \varepsilon_L|_0 \omega L + k \tilde{q}_j n_j &= 0 \\
(\omega - i N_1) \tilde{q}_1 + k [\tilde{\gamma}]_0 n_1 T + (k [\kappa])_0 n_1 - i N_3) L &= 0 \\
(\omega - i N_1) \tilde{q}_2 + k [\tilde{\gamma}]_0 n_2 T + (k [\kappa])_0 n_2 - i N_3) L &= 0 \\
(\omega - i N_1) \tilde{q}_3 + k [\tilde{\gamma}]_0 n_3 T + (k [\kappa])_0 n_3 - i N_3) L &= 0 \\
(\omega - i N_2) \tilde{L} + k [\tilde{\gamma}]_0 n_j + i N_4 n_j &= 0
\end{align*}
\]

(3.12)

In the following, the subscript 0, which denotes quantities referring to the unperturbed state (3.8), will be dropped out.

This system possesses nontrivial solutions if and only if its determinant vanishes. Imposing this condition, in the case \( \mathbf{n} = (1, 0, 0) \), i.e. when the wave is collinear to the heat flux \( \mathbf{q} \), one obtains

\[
\omega^2 = k^2 \left[ \tilde{V}_2^2 (1 - \rho \varepsilon_L N) + \nu \kappa \right] + N_1 N_2 - i \omega (N_1 + N_2) + i k \tilde{V}_2^2 N_2
\]

(3.13)

\[
+ i k \left[ \kappa - \tilde{V}_2^2 \rho \varepsilon_L N_4 - \nu_0 N_3 \right] + N_3 N_4
\]
while, in the case \( \mathbf{n} = (0, 0, 1) \), i.e. when the wave is orthogonal to the heat flux \( \mathbf{q} \), one obtains

\[
\omega^2 = k^2 \left[ \bar{V}_2^2 (1 - \rho \varepsilon_L V) + \nu \chi \right] + N_1 N_2 - i\omega(N_1 + N_2)
\]

\[+ \frac{k^2}{\alpha} \bar{V}_2^2 \left( i N_2 - \frac{N_1 N_3}{\alpha + i \alpha L} \right) + N_3 N_4. \tag{3.14}
\]

In both equations (3.13) and (3.14), we have made the position

\[
\bar{V}_2 = \sqrt{\frac{\zeta}{\rho c v}}. \tag{3.15}
\]

The quantity \( \bar{V}_2 \), in the absence of vortices \( (\zeta = \zeta^*) \), coincides with the usual velocity of the second sound \( V_2^* \), mentioned in subsection 2.1 and, in the presence of vortices, includes a positive contribution proportional to \( L \), which shows that the speed of the waves increases when \( L \) increases. In the first term on the right-hand side of (3.13) -the term related with the speed of waves-, this effect is further enhanced when the vortex tangle is deformed by the second sound.

### 3.2 Inhomogeneous anisotropic superfluid turbulence

Experiments [18] and numerical simulations [19, 20, 21] show that the vortices in the tangle tend moving outwards in a plane orthogonal to the external heat flux, thus leading to anisotropy in the vortex line distribution with vortices concentrated in planes orthogonal to the external heat flux. If the turbulence is produced in a cylindrical channel, it is possible to suppose the isotropy in planes orthogonal to the external heat flux. In this case, there are many segments of vortex lines in one direction than in the opposite direction, and one has, if one chooses the external heat flux \( \mathbf{q}_{\text{ext}} \) along the first axis, the following term:

\[
P_{ij}^o = \frac{1}{2} \kappa L B \begin{pmatrix} 2a & 0 & 0 \\ 0 & 1 - a & 0 \\ 0 & 0 & 1 - a \end{pmatrix}.
\tag{3.16}
\]

Here \( a \) is the parameter characterizing the anisotropy of the tangle. Indeed one has:

\[
< s_x'^2 > = 1 - 2a, \quad < s_y'^2 > = < s_z'^2 > = a, \tag{3.17}
\]

and the production term in the equation of the heat flux assumes the expression:

\[
\sigma^q = -\frac{1}{2} \kappa L B \begin{pmatrix} 2aq_1 \\ (1 - a)q_2 \\ (1 - a)q_3 \end{pmatrix}.
\tag{3.18}
\]

### 3.3 Second sound propagation in anisotropic turbulent superfluids

The aim of this subsection is to investigate the propagation of plane harmonic waves in turbulent superfluids in the presence of an anisotropic vortex tangle and to calculate the speeds of propagation and the attenuation coefficients of these waves.

A stationary solution of system (3.1), with \( \sigma^q \) and \( \sigma^L \) expressed by equations (3.18) and (3.6) is:
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\[
T = T_0 - \gamma_1 \frac{L_0 q_1^0}{c} x_1, \quad q = q_0 = (q_1^0, 0, 0), \quad L = L_0 = (\gamma_1 q_1^0)^2, \quad (3.19)
\]

where \( \gamma_1 = \alpha / \beta \), \( \gamma_1 = a \kappa B \) and with \( q_1^0 > 0 \).

To study the propagation of harmonic waves in this turbulent state, in a neighborhood of this solution, we substitute in (3.1) \( \sigma_f \) and \( \sigma_L \) with the approximate expressions:

\[
\begin{align*}
\sigma_1^q & \simeq \gamma_1^{1/2} q_1 \left[ -q_1^0 (L - L_0) - L_0 q_1 \right], & \text{with} \quad \gamma_1 = a \kappa B \\
\sigma_2^q & \simeq -\gamma_1 L_0 q_2, & \text{with} \quad \gamma_2 = \frac{1}{2} (1 - a) \kappa B \\
\sigma_3^q & \simeq -\gamma_1 L_0 q_3, & \text{with} \quad \gamma_3 = \frac{1}{2} (1 - a) \kappa B \\
\sigma_L & \simeq -\frac{1}{2} \tilde{B} L_0 (L - L_0) + \tilde{a} L_0^{3/2} (q_1 - q_1^0),
\end{align*}
\]

obtaining the system:

\[
\begin{align*}
\frac{\partial T}{\partial t} + A_2 \frac{\partial T}{\partial x_1} + \frac{\partial T}{\partial x_2} + \frac{\partial T}{\partial x_3} &= -\gamma_1 q_1^0 (L - L_0) - \gamma_1 L_0 q_1 \\
\frac{\partial q_1}{\partial t} + \frac{\partial T}{\partial x_1} + \frac{\partial T}{\partial x_2} &= -\gamma_2 L_0 q_2 \\
\frac{\partial q_2}{\partial t} + \frac{\partial T}{\partial x_2} + \frac{\partial T}{\partial x_3} &= -\gamma_3 L_0 q_3 \\
\frac{\partial T}{\partial t} + \frac{\partial q_1}{\partial x_2} + \frac{\partial q_1}{\partial x_3} &= -\frac{1}{2} \tilde{B} L_0 (L - L_0) + \tilde{a} L_0^{3/2} (q_1 - q_1^0)
\end{align*}
\]

where \( \epsilon_L = \epsilon_V = \partial \epsilon / \partial L, \epsilon_T = \epsilon_V = \partial \epsilon / \partial T \) and we have made the following position:

\[
A_2 = \frac{1}{\rho \epsilon_T} (1 - \rho \epsilon_L \nu) \quad (3.22)
\]

In the following we will study the wave propagation in liquid helium in two cases of physical interest: when the wave is propagating in a direction parallel to the heat flux, and when the wave propagation direction is orthogonal to it.

First case: \( \mathbf{n} \) parallel to the heat flux \( \mathbf{q} \)

We analyze here the case in which the unit vector \( \mathbf{n} \) orthogonal to the wave front is parallel to the heat flux direction. Without prejudice one can choose \( \mathbf{n} = (1, 0, 0) \) as unit vector in the direction of the wave propagation, and \( \mathbf{t}_1 = (0, 1, 0) \) and \( \mathbf{t}_2 = (0, 0, 1) \) as unit vectors tangent to the wave front.

Consider the propagation of harmonic plane waves, seeking for solutions of equations (3.21) of the form

\[
\begin{align*}
T &= T_0 - \gamma_1 \frac{L_0 q_1^0}{c} x_1 + \tilde{T} e^{(k x_1 - \omega t)} \\
q_j &= q_j^0 + \tilde{q}_j e^{(k x_1 - \omega t)} \\
L &= L_0 + \tilde{L} e^{(k x_1 - \omega t)}
\end{align*}
\]
where \( q_0^1 = (q_1^0, 0, 0) \), \( k = k_s + ik_t \) is the wave-number, \( \omega = \omega_e + i\omega_r \) the frequency. Inserting (3.23) in the equations (3.21), the following homogeneous algebraic system for the small amplitudes is obtained:

\[
\begin{cases}
-\omega T + (k [A_2]_0 - i \frac{\Delta \omega}{\gamma} L_0^{3/2}) \bar{q}_1 + i \frac{1}{2} \frac{\omega}{\gamma} \bar{L}_0 L = 0 \\
(-\omega - i \gamma q_1^0) \bar{q}_1 + k [\zeta]_0 T + (k [\chi]_0 - i \gamma q_1^0) \bar{L} = 0 \\
(-\omega - i \gamma q_1^0) \bar{q}_2 = 0 \\
(-\omega - i \gamma q_1^0) \bar{q}_3 = 0 \\
(-\omega - i \gamma q_1^0) \bar{L} + (k [\nu]_0 + i \alpha L_0^{3/2}) \bar{q}_1 = 0
\end{cases}
\] (3.24)

In (3.24) the subscript 0 denotes quantities referring to the unperturbed state (3.19). In what follows, for the sake of simplicity, this subscript will be neglected. Now, we look for a solution of the small amplitudes system (3.24). One deduces immediately that waves longitudinal and transversal in velocity and heat flux propagate independently. We consider only the longitudinal modes, corresponding to the equations (3.24a), (3.24b) and (3.24c). Imposing the vanishing of the determinant of this system, one obtains:

\[
-\omega(\omega + i \gamma) \left( \omega + \frac{1}{2} \bar{B} L_0 \right) + \omega \left( k \chi - i \gamma q_1^0 \right) + k \xi \left( \omega + \frac{1}{2} \bar{B} L_0 \left( k A_2 - i \alpha L_0^{3/2} \right) + \frac{1}{2} \frac{\omega}{\gamma} \bar{B} L_0 \left( k \chi - i \gamma q_1^0 \right) \right) = 0,
\] (3.25)

To study the solutions of this equation we will use a perturbation method. First we will determine the solutions of the dispersion relation (3.25) and the corresponding modes under the hypothesis \( L_0 = 0 \). After the solutions of the unperturbed system are determined, the usual procedure of the perturbation theory will be followed in order to obtain a hierarchy of equations for the various orders in the small parameter \( L_0^{1/2} = \frac{a}{b} q_1^0 \).

Setting \( L_0^{1/2} = \frac{a}{b} q_1^0 = 0 \) in dispersion relation (3.25), we get the equation:

\[
\omega \left( \omega^2 - k^2 (\xi A_2 + \nu \chi) \right) = 0,
\] (3.26)

which corresponds to the unperturbed solutions \( \omega_5 = 0 \) and:

\[
\left[ \frac{\omega_2^{(0)}}{\omega_5} \right] = \pm \sqrt{\nu_2} = \pm \sqrt{|V_2|^2 (1 - \rho \epsilon \nu)} + \nu \chi,
\] (3.27)

\( k_s = 0 \).

The solution \( \omega_2^{(0)} \) is the unperturbed second sound velocities (in the absence of vortex lines). Note that in this model \( \omega_2^{(0)} \) is different from the second sound speed determined in [15] and in [5], where line density \( L \) was considered a dependent field. The solution \( \omega_2^{(0)} = 0 \) corresponds to a new mode, which was not present in the previous models.

We perturb the solution \( \omega_2^{(0)} = \pm k, V_2, k_{2s}^{(0)} = 0 \), corresponding to the second sound, putting: \( \omega_2 = \omega_2^{(0)} + \delta \omega_2 \) and \( k_{2s} = k_{2s}' \). We will suppose here that the line density \( L \) is small, so that \( L_0^{1/2} \) can be
considered as a small parameter, and $\omega'$ and $k'_s$ are also small corrections of the same order, in such a way that their products can be neglected. So we have:

$$\omega_2 \simeq \omega_2^{(0)} + \omega_2^{(1)} L_0^{1/2},$$

$$k_{2s} \simeq h_{2s}^{(1)} L_0^{1/2}.$$  \hspace{1cm} (3.28)

In this case, because we have in mind to obtain an approximate expression for the second sound, we suppose $w^2 = w^r$ a real frequency, and $k = k_r + ik_s$ a complex wave-number. We obtain $\omega_2^{(1)} = 0$ and

$$w^2 \simeq w^r_{(0)}^2 + v \phi_{(1)}^2 L_1 = 20;$$

$$k_{2s} \simeq \frac{\gamma \phi^2}{20} L_0^{1/2}.$$  \hspace{1cm} (3.29)

**Second case: n orthogonal to the heat flux q**

Now, we analyze the case in which the unit vector $n$ orthogonal to the wave front is orthogonal to the heat flux direction.

Choosing $q = (1, 0, 0)$, $n = (0, 0, 1)$, $t_1 = (1, 0, 0)$ and $t_2 = (0, 1, 0)$, we seek for solutions of equations (3.21) of the form

$$\begin{align*}
T &= T_0 - \gamma \phi_{(2)} L_0^0 + T e^{i(kx_3 - \omega_t)}, \\
q_j &= q_j^{(0)} + \bar{q}_j e^{i(kx_3 - \omega_t)}, \\
L &= L_0 + L \phi_{(2)} e^{i(kx_3 - \omega_t)},
\end{align*}$$

(3.30)

Inserting (3.30) in the equations (3.21), the following homogeneous algebraic system for the small amplitudes is obtained:

$$\begin{align*}
-\omega T + k [A_2]_0 \bar{q}_3 + i \frac{\gamma \phi_{(2)}}{\xi} \left( \frac{1}{2} \bar{L} L_0 - \alpha L_0^{3/2} \bar{q}_1 \right) &= 0, \\
(-\omega - \gamma_{\parallel} L_0) \bar{q}_1 - \gamma_{\parallel} q_0 \bar{L} &= 0, \\
(-\omega - \gamma_{\parallel} L_0) \bar{q}_2 &= 0, \\
(-\omega - \gamma_{\parallel} L_0) \bar{q}_3 + k [\zeta]_0 \bar{T} + k [\chi]_0 \bar{L} &= 0, \\
(-\omega - i \frac{1}{2} \bar{L} L_0) \bar{L} + k [\psi]_0 \bar{q}_3 + i \bar{L} L_0^{3/2} \bar{q}_1 &= 0.
\end{align*}$$  \hspace{1cm} (3.31)

One sees that, in this case, vibrations of $q$ orthogonal and parallel to the wave front are coupled. Consider now the equations (3.31a), (3.31b), (3.31d) and (3.31e) of the small amplitude system. This system admits nontrivial solutions if and only if its determinant vanishes. Imposing this condition one obtains the following fourth degree equation in $\omega$:
\[ \begin{aligned} &\omega \left[ (\omega + i\gamma_{\parallel}L_0) \left( \omega + i\gamma_{\perp}L_0 \right) \right. \\
&\left. - v_{\parallel}^2 k^2 - \gamma \bar{a} q_1^0 \tilde{L}_0 \left( \omega + i\gamma_{\perp}L_0 \right) \right] \\
&+ k\zeta(\omega + i\gamma_{\parallel}L_0) \left[ kA_2 \left( - \omega - i\tilde{B}L_0 \right) - kv_{\perp}^2 \tilde{B}L_0 \frac{\xi_2}{\xi_1} \right] \\
&= \gamma \bar{a} q_1^0 \tilde{L}_0 \frac{1}{2} k^2 \zeta \left( A_2 + v_{\parallel}^2 \right) = 0 \end{aligned} \] (3.32)

As in the previous case, to study the solutions of this equation we will use a perturbation method. The solutions of the dispersion relation (3.32) under the hypothesis \( L_0 = 0 \) are \( w(0) = 0 \) and \( \omega(0) = 0 \) as expressed in equation (3.27).

In this case the first order approximations are zero. Indeed, perturbing the solution \( \omega(0) = \pm k, \sqrt{\zeta A_2 + v_{\perp}} k(2) s = 0 \) with:

\[ \begin{aligned} \omega_2 &\simeq \omega(0) + \sigma_{(1)}^2 \frac{1}{2} + \sigma_{(2)}^2 L_0, \\
&\omega_2 \simeq h_{(2) s} L_0 \end{aligned} \] (3.33)

we obtain \( \sigma_{(1)}^2 = 0, h_{(1) s} = 0, \sigma_{(2)}^2 = 0 \) and

\[ \begin{aligned} \frac{\omega_2}{k} &\simeq \pm V_2, \\
k_{(2) s} &\simeq \frac{V_2^2 \left( A_2 + v_{\parallel}^2 \right)}{2V_2^2} \tilde{L}_0. \end{aligned} \] (3.34)

4 Conclusions

In this work, previous results on the propagation of second sound in superfluid liquid \(^4\)He both in absence and in the presence of a turbulent vortex tangle are examined. It is shown that in the absence of the vortex tangle (laminar flow) the dissipative shear and bulk thermal forces, that are described using the thermal stress (2.3), modify the amplitude of the heat wave, but not its velocity. Instead, in the presence of a turbulent vortex tangle both velocity and amplitude of the second sound are modified. In particular, in the case of the anisotropy of the tangle, the expansion in terms of the small parameter \( L^{1/2} \) of the attenuation coefficient of the second sound starts with the first order when the wave propagates collinear to the heat flux, i.e. is proportional to \( L^{1/2} \), but starts with second order terms, i.e. is proportional to \( L \), when the wave propagates orthogonal to the heat flux.

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References


