Natural transform for distribution and Boehmian spaces

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Abstract. The Natural transform ($N$ transform) is employed for distribution spaces $\mathcal{D}'$, the dual of the space $\mathcal{D}$, the basic theorems and properties are proved for this space. The integrable Boehmian space is also defined for the Natural transform.

1 Introduction

The theory of distribution spaces (i.e. generalized functions) and its properties are developed by Schwartz [14], which are known to have many applications. One among such is the use of various types of integral transforms to the distribution spaces [1, 7, 8, 13, 14, 18, 19]. The paper has three sections for explicit comprehension. In Section 1, in what follows below, definitions and properties of the Natural transform are given. In Section 2 we have proved theorems for the Natural transform on certain distribution spaces. Section 3 deals with the result that studies the Natural transform for integrable Boehmians.

Numerous integral transforms, such as Fourier, Laplace, Hankel, and many more are used to solve differential and integral equations. The new transform $N$-transform is derived by Khan and Khan [5] its properties and applications are discussed. The inverse and additional properties of the $N$-transform are given by Belgacem et al. [4].

The real function $f(t) > 0$ and $f(t) = 0$ for $t < 0$, is section wise continuous, of exponential order, and defined in the set $A$ by
\[ A = \{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{t/\tau_2}; t \in (-1)^j \times [0, \infty) \}, \]

where \( M \) is a constant of finite number and \( \tau_1 \) and \( \tau_2 \) may be finite or infinite.

The Natural transform (i.e. the \( N \) transform) \( R(s, u) \) of the function \( f(t) \) for all \( t \geq 0 \) is given by [4, 5]

\[
N^+[f(t)] = R(s, u) = \frac{1}{u} \int_0^\infty e^{-u} f(ut) dt \quad , \quad s > 0, u > 0
\]

(1.1)
i.e.

\[
R(s, u) = \frac{1}{u} \int_0^\infty e^{-\frac{u}{t}} f(t) dt
\]

(1.2)

where \( t, u \) are time variable and \( s \) is the frequency variable.

This can also be written in discrete form as [4, p.102, Eqn. (2.14)]

\[
N^+[f(t)] = R(s, u) = \sum_{n=0}^{\infty} n! a_n u^n s^{n+1}
\]

(1.3)

The inverse Natural transform is defined by [4, p.101, Eqn. (2.7)], see also [15]

\[
N^{-1}[R(s, u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} u R(s, u) ds
\]

(1.4)

The duality relation between the Natural-Laplace and Natural-Sumudu transforms is suggested by

\[
R(s, u) = \frac{1}{u} F \left( \frac{s}{u} \right) \quad ; \quad R(s, u) = \frac{1}{s} G \left( \frac{u}{s} \right)
\]

(1.5)

where \( R(s, u) \) denote the Natural transform , \( F(s) \) is the Laplace transform and \( G(u) \) is the Sumudu transform. The Sumudu transform, its properties and applications can be seen in [2, 3, 7, 8, 16, 17]. We mention couple of properties of the Natural transform, proofs may be seen in [4, 5]

1. **Natural transform of derivative** : The derivative of \( f(t) \) with respect to \( t \), and \( n \) th order derivative of \( f(t) \) with respect to \( t \) are, respectively, defined by

\[
N^+[f'(t)] = R_1(s, u) = \frac{s}{u} R(s, u) - \frac{f(0)}{u}
\]

(1.6)

and

\[
N[f^{(n)}(t)] = R_n(s, u) = \frac{s^n}{u^n} R(s, u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} f^{(k)}(0)
\]

(1.7)

2. When \( f(t) = \delta(t) \) (the Dirac delta function), the Natural transform becomes

\[
N^+[\delta(t)] = R(s, u) = \frac{1}{u}
\]

(1.8)

more such for other values are tabulated in [4, 5].

3. **Convolution Theorem** : If \( F(s, u) \) and \( G(s, u) \) are Natural transforms of the functions \( f(t) \) and \( g(t) \) defined in set \( A \), then the convolution is given by
\[ N^+[(f \ast g)(t)] = u F(s, u)G(s, u) \; . \] (1.9)

4. **Linearity property**: If \( \alpha, \beta \) are any constants and \( f(t) \) and \( g(t) \) are functions, then

\[
N^+[\alpha f(t) + \beta g(t)] = \alpha F(s, u) + \beta G(s, u) \; .
\] (1.10)

**Theorem 1.1.** [5, p. 127]: If \( f(t) \) is sectionally continuous in every finite interval \( 0 \leq t \leq K \) and of exponential order \( \gamma \) for \( t > K \) then its Natural (or N) transform \( R(s, u) \) exists for all \( s > \gamma, u > \gamma \).

\[
\int_{-\infty}^{\infty} e^{-st}f(ut)dt \leq \int_{-\infty}^{\infty} f(ut)|dt| \leq \int_{0}^{\infty} e^{-st}Me^{bt}dt = \frac{M}{s - \gamma u} \; .
\]

**2 Distributional natural transform**

In what follows, with regard to [18] and by virtue of terminologies used therein, we brief the description of testing function space and functionals and employ them to study the Natural transform on distribution spaces.

The space of testing functions, which is denoted by \( \mathcal{D} \), consists of all complex valued functions \( \varphi(t) \) that are infinitely smooth and vanishes outside some finite interval. A sequence of testing functions \( \{\varphi_\upsilon(t)\}_{\upsilon=1}^{\infty} \) is said to converge in \( \mathcal{D} \) if \( \varphi_\upsilon(t) \) are in \( \mathcal{D} \), if they are all zero outside some fixed interval \( I \), and if for every fixed non-negative integer \( k \), the sequence \( \{\varphi^{(k)}_\upsilon(t)\}_{\upsilon=1}^{\infty} \) converges uniformly for \( -\infty < t < \infty \).

The functionals that assign a complex number to every member of \( \mathcal{D} \) is denoted by \( \langle f, \varphi \rangle \) which have two properties, linearity and continuity.

**Linearity**: The functional \( f \) on \( \mathcal{D} \) is said to be linear if testing functions \( \varphi_1 \) and \( \varphi_2 \) and any complex number \( \alpha \), form relations

\[
\begin{align*}
\langle f, \varphi_1 + \varphi_2 \rangle &= \langle f, \varphi_1 \rangle + \langle f, \varphi_2 \rangle \\
\langle f, \alpha \varphi_1 \rangle &= \alpha \langle f, \varphi_1 \rangle
\end{align*}
\] (2.1)

**Continuity**: A functional \( f \) on \( \mathcal{D} \) is said to be continuous if for any sequence of testing functions \( \{\varphi_\upsilon(t)\}_{\upsilon=1}^{\infty} \) converges to \( \varphi(t) \) in \( \mathcal{D} \). In other words, if \( f(t) \) is a locally integrable function, then the distribution \( f \) the convergent integral \( (f(t) \text{ in } \mathcal{D}^{'}) \), defined by

\[
\langle f, \phi \rangle = \langle f(t), \phi(t) \rangle = \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)dt \; ,
\] (2.2)

shows that the function \( f(t) \) generates a distribution, or we say, \( f(t) \) is in \( \mathcal{D}' \) and \( \phi(t) \) belongs to \( \mathcal{D} \), where \( \mathcal{D} \) and \( \mathcal{D}' \) denote, respectively, testing function space and its dual.
A continuous linear functional on the space \( \mathcal{D} \) is a distribution. The space of all such distributions is denoted by \( \mathcal{D}' \) (\( \mathcal{D}' \) is called the dual (or conjugate) space of \( \mathcal{D} \)).

The locally integrable function \( f(t) \) is absolutely integrable over \( 0 < t < \infty \)

\[
\int_0^{\infty} |f(t)| \, dt < \infty. \tag{2.3}
\]

The Natural transform \( R(s,u) \) of \( f(t) \) is a function for all \( t \geq 0 \), that is defined by

\[
N^+[f(t)] = R(s,u) = \frac{1}{u} \int_0^{\infty} e^{-\frac{s}{u}} f(t) \, dt
\]

By virtue of (13) and Theorem 1, the Natural transform \( N^+[f(t)] \) is bounded for all \( t \) and uniformly continuous. Moreover, we can also have

\[
|R(s+\eta) - R(s,u)| = \frac{1}{u} \left[ e^{-(\frac{s+\eta}{u}) t} f(t) dt - \frac{1}{u} \int_0^{\infty} e^{-\frac{s}{u}} f(t) \, dt \right] \\
\leq \left| f(t) \right| \left| \frac{1}{u} e^{-\eta t} (e^{-\eta t} - 1) \right| dt \\
\leq \varepsilon
\]

It may not be out of place to remark that, if the locally integrable functions \( f(t) \) and \( g(t) \) are absolutely integrable over \( 0 < t < \infty \) and if the Natural transforms \( F(s,u) \) and \( G(s,u) \) are equal everywhere, then \( f(t) = g(t) \) almost everywhere. Further, in what follows, we obtain the Parseval relation (or equation) to enable us to study the Natural transform and its inverse in the sense of distributions (i.e., distributional \( N^- \) transform).

**Theorem 2.1.** If locally integrable functions \( f(t) \) and \( g(t) \) are absolutely integrable over \( 0 < t < \infty \), then

\[
\int_0^{\infty} F(s,u) G(s,u) \, ds = \int_0^{\infty} \frac{1}{u} f(t) g(-t) \, dt . \tag{2.4}
\]

**Proof:** Since \( F(s,u) \) and \( G(s,u) \) are bounded and continuous for all \( t \), therefore (14) converges. Moreover,

\[
\int_0^{\infty} F(s,u) G(s,u) \, ds = \frac{1}{u} \int_0^{\infty} e^{-\frac{s}{u}} f(t) \, dt \int_0^{\infty} G(s,u) \, ds \\
= \frac{1}{u} \int_0^{\infty} f(t) \, dt \int_0^{\infty} G(s,u) e^{-\frac{s}{u}} \, ds \\
= \frac{1}{u} \int_0^{\infty} f(t) g(-t) \, dt, \quad \text{from (4)}.
\]

Since the above integral is absolutely integrable, therefore

\[
\int_0^{\infty} F(s,u) G(s,u) \, ds = \int_0^{\infty} \frac{1}{u} f(t) g(-t) \, dt .
\]

Further, we consider \( g(t) = f^*(\cdot) \) such that
\[ \int_0^\infty F(s,u)F^*(s,u)ds = \frac{1}{u} \int_0^\infty f(t)f^*(t)dt \]
i.e.
\[ \int_0^\infty |F(s,u)|^2 dv = \int_0^\infty \frac{1}{u} |f(t)|^2 dt . \]

Thus, the Parseval relation for the Natural transform can be written as
\[ \|F\| = \frac{1}{u} \|f\| . \tag{2.5} \]

This proves the theorem.

We can state from Theorem 2 that \( f \in L_p \) (or square integrable when \( p = 2 \)), indeed, the distributional Natural transform \( R(s,u) \) of the function \( f(t) \in \mathcal{D}' \) can be written as
\[ N^+[f(t)] = R(s,u) = \left\langle f(t), \frac{1}{u} e^{-\frac{u}{s}} \right\rangle . \tag{2.6} \]

**Theorem 2.2.** If \( f \) and \( g \) are Natural transformable distributions in \( \mathcal{D}' \) and \( F(s,u) = N^+\{f(t)\}; G(s,u) = N^+\{g(t)\} \), then \( f \ast g \) is a Natural transformable distribution in \( \mathcal{D}' \)
\[ N^+[(f \ast g)(t)] = u \ F(s,u) G(s,u) . \tag{2.7} \]

**Proof:** Since the functions \( f(t) \) and \( g(t) \) are bounded and analytic, \( \frac{1}{u} e^{-\frac{u}{s}} \in \mathcal{D} \), we have
\[
N^+[(f \ast g)(t)] = \left\langle (f \ast g)(t), \frac{1}{u} e^{-\frac{u}{s}} \right\rangle \\
= \left\langle f(t), \left\langle g(\tau), \frac{1}{u} e^{-\frac{u(s+\tau)}{s}} \right\rangle \right\rangle \\
= u \left\langle f(t), \frac{1}{u} e^{-\frac{u}{s}} \right\rangle \left\langle g(\tau), \frac{1}{u} e^{-\frac{u}{s}} \right\rangle \\
= u \ F(s,u) G(s,u).
\]

The theorem is completely proved.

Since \( f(t) \in \mathcal{D}', N^+\{f(t)\} = F(s,u) \), the operation formulae discussed in Section 1 and those in \([4, 5]\) are in \( \mathcal{D}' \).

### 3 Natural Transform and Boehmian Spaces

The general construction of Boehmians is given in \([10, 11]\) which when applied to different function spaces, various Boehmian spaces result. The term Boehmian is used for all objects by an abstract algebraic construction, similar to that of the field of quotients. Let \( G \) be an additive commutative semigroup and \( S \subseteq G \), is a sub-semigroup, which has a mapping \( \ast \) from \( G \times S \) to \( G \) such that
(i) if \( \delta, \eta \in S \), then \( (\delta \ast \eta) \in S \) and \( \delta \ast \eta = \eta \ast \delta \)
(ii) if \( \alpha \in G, \delta, \eta \in S \), then \( (\alpha \ast \delta) \ast \eta = \alpha \ast (\delta \ast \eta) \)
(iii) if \( \alpha, \beta \in G, \delta \in S \), then \( (\alpha + \beta) * \delta = (\alpha * \delta) + (\beta * \delta) \).

The member of the class \( \Delta \) of sequence from \( S \) are called the delta sequence, which satisfies the following:

(i) if \( \alpha, \beta \in G, (\delta_n) \in \Delta \) and \( (\alpha * \delta_n) = (\beta * \delta_n) \), for all \( n \), then \( \alpha = \beta \) in \( G \).

(ii) if \( (\delta_n), (\varphi_n) \in \Delta \), then \( (\delta_n * \varphi_n) \in \Delta \).

We consider a quotient of the sequence \( f_n/\varphi_n \), numerator of which belongs to \( G \) and the denominator is a delta sequence, and

\[
f_m * \varphi_n = f_n * \varphi_m, \quad \text{for all } m, n \in N. \tag{3.1}
\]

Two quotients of sequences \( f_n/\varphi_n \) and \( g_n/\psi_n \) are said to be equivalent if

\[
f_n * \psi_n = g_n * \varphi_n, \quad \text{for all } n \in N. \tag{3.2}
\]

The equivalence classes, thus obtained, are called Boehmians, the space of all of which bears the notation \( B \) and an element of which is written as \( x = f_n/\varphi_n \). If we consider \( G \) to be the set of all locally integrable functions on \( \mathbb{R} \), then the Bohmian space \( B \) is called the space of locally integrable Bohmian \( B_{L_1} \), which has properties of addition, multiplication by scalar, convolution in a convolution algebra \([12]\).

Let \( L_1 \) is the space of complex valued Lebesgue integrable functions on the real line \( \mathbb{R} \), norm of which in \( L_1 \) is \( ||f|| = \int_{\mathbb{R}} |f(x)| \, dx \). If \( f, g \in L_1 \), then the convolution product \( f * g \), i.e., \( (f * g)(x) = \int_{\mathbb{R}} f(u)g(x-u) \, du \) is an element of \( L_1 \) and that \( ||f * g|| \leq ||f|| \cdot ||g|| \).

A sequence of continuous real functions \( \delta_n \in L_1 \) will be called a delta sequence if

(i) \( \int_{\mathbb{R}} \delta_n(x) \, dx = 1 \), \( \forall n \in N \)

(ii) \( ||\delta_n|| < M \), for some \( M \in \mathbb{R} \) and all \( n \in N \), and

(iii) \( \lim_{n \to \infty} \int_{|x| > \varepsilon} |\delta_n(x)| \, dx = 0 \), for each \( \varepsilon > 0 \).

For convergence of Bohmians, see \([11]\) and refer to \([12]\) for the properties of integrable Boehmians.

A function \( f \in L_1 \) can be identified with the Boehmian \( [f * \delta_n/\delta_n] \), where \( (\delta_n) \) is a delta sequence. If \( F = [f_n/\varphi_n] \), then \( f * \delta_n = f_n \). Therefore, \( f * \delta_n \in L_1, \forall n \in N \).

A sequence of Boehmians \( F_n \) is \( \Delta \)-convergent to a Boehmian \( F \) if there exists a delta sequence \( (\delta_n) \) such that \( (F_n - F) * \delta_n \in L_1 \), for every \( n \in N \) and \( ||F_n - F) * \delta_n|| \to 0 \), as \( n \to \infty \).

A sequence of Boehmians \( F_n \) is \( \delta \)-convergent to a Boehmian \( F \) if there exists a delta sequence \( (\delta_n) \) such that \( F_n * \delta_k \in L_1 \) and \( F * \delta_k \in L_1 \), for every \( n, k \in N \) and \( ||F_n - F) * \delta_k|| \to 0 \), for each \( k \in N \).

The theory of Boehmian spaces of different classes are associated with different types of integral transforms, for details see \([1, 6, 7, 8, 9, 12]\). This section deals with the investigation of Natural transform for integrable Boehmian.

**Lemma 3.1.** If \( [f_n/\delta_n] \in B_{L_1} \), then the sequence

\[
N^+ \left[ f_n(t) \right] = R(s, u) = \frac{1}{u} \int_0^\infty e^{-\frac{u}{2}t} f_n(t) \, dt
\]

converges uniformly on each compact set in \( \mathbb{R} \).
**Proof.** If \((\delta_n)\) is a delta sequence, then \(N^+(\delta_n)\) converges uniformly on each compact set to a constant function \(\frac{1}{n}\). Therefore, for each compact set \(K\), \(N^+(\delta_k) > 0\) on \(K\), and for almost all \(k \in K\). Further,

\[
N^+(f_n) = N^+(f_n) \frac{N^+(\delta_k)}{N^+(\delta_k)} = \frac{1}{u} \frac{N^+(f_n \ast \delta_k)}{N^+(\delta_k)}
\]

that is,

\[
\frac{N^+(f_k \ast \delta_n)}{u N^+(\delta_k)} = \frac{N^+(f_k)}{N^+(\delta_k)} N^+(\delta_n) \quad \text{on } K.
\]

Thus, the lemma is proved.

By virtue of the Lemma 1, the Natural transform of an integrable Boehmian \(F = [f_n/\delta_n]\) can be defined as the limit of \(N^+(f_n)\) in the space of continuous functions on \(\mathbb{R}\). Thus, the Natural transform of an integrable Boehmian is a continuous function.

**Theorem 3.1.** If \([f_n/\delta_n] \in B_{L_1}\), then the sequence \(N^+(f_n)\) converges in \(D'\). Moreover, if \([f_n/\delta_n] = [g_n/\delta_n]\), then \(N^+(f_n)\) and \(N^+(g_n)\) converges to the same limit of Natural transform for integrable Boehmian.

**Proof:** Let \([f_n/\delta_n], [g_n/\delta_n] \in B_{L_1}\) such that \([f_n/\delta_n] = [g_n/\delta_n]\)

\(f_n \ast \delta_n = g_n \ast \delta_n\)

Applying convolution of Natural transform on both the sides, we have

\[
uN^+(f_n)N^+(\delta_n) = u N^+(g_n)N^+(\delta_n)
\]

\[
N^+[f_n/\delta_n] = N^+[g_n/\delta_n]
\]

This proves the theorem.

**4 Conclusion**

This paper is suddenly brought sketches together into a single whole, due to a new avenue shown in Belgacem and Silambarasan [4] and [15]. The Natural transform (or \(N\)-transform) has been studied explicitly for the Schwartz distributions and integrable Boehmian. To the extent possible (and thought reasonable), part of this paper defines terminologies employed. Parseval relation for the Natural transform is obtained including its inverse in order to enable a possible formula for the distributional Natural transform.

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