Soliton perturbation theory for complex modified KdV equation in plasmas with full nonlinearity

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Abstract. The complex modified KdV equation is studied in presence of perturbation terms with full nonlinearity. The soliton perturbation theory is applied to obtain the adiabatic dynamics of the soliton amplitude and frequency. The slow change in the soliton velocity is also obtained in presence of these perturbation terms.

1 Introduction

The soliton perturbation theory (SPT) is a very important tool in the area of Theoretical Physics that is applied to obtain the adiabatic variation in the soliton parameters that include the soliton amplitude, width, energy, linear and angular momentum and many others. This leads to analytical expressions in the slow dynamics of these parameters when the perturbation terms are turned on. These slow dynamics leads to a lot of information in the perturbed soliton when they are propagating through any nonlinear media [1-10].

The complex modified KdV (cmKdV) equation that arises in the context of Plasma Physics, will be studied in this paper in presence of perturbation terms that are considered with full nonlinearity [5, 6]. The SPT will be applied to cmKdV equation to obtain the adiabatic dynamic of the soliton amplitude and frequency only. Subsequently, the slow change in the soliton velocity will also be determined.

2 Mathematical analysis

The cmKdV equation that will be studied in this paper is given by [1-10]
where \( a \) and \( b \) are constants. The dependent variable \( q \) represents the wave profile in plasmas while the independent variables \( x \) and \( t \) that are the spatial and temporal variables respectively. The first term in (2.1) represent the evolution term, while the coefficient of \( a \) is the nonlinear term and finally the coefficient of \( b \) is the dispersion term.

This equation has already been studied earlier [1-10]. Integration of (2.1) was carried out using several techniques. The 1-soliton solution of (2.1) is given by [1, 2]

\[
q(x,t) = A \text{sech}[B(x - vt)] e^{i(-\kappa t + \omega t + \Theta)}
\]  

(2.2)

where \( A \) is the soliton amplitude while \( B \) is the inverse width of the soliton and \( v \) is the soliton velocity. Finally, in the phase component, \( \kappa \) represents the soliton frequency, \( \omega \) is the soliton wave number and \( \Theta \) is the phase constant. The amplitude-width relation is given by [1, 2]

\[
B = A \sqrt{\frac{ab}{6b}}
\]  

(2.3)

and the velocity of the soliton is given by

\[
v = b \left( B^2 - 3\kappa^2 \right)
\]  

(2.4)

and finally the wave number is

\[
\omega = b\kappa \left( 3B^2 - \kappa^2 \right)
\]  

(2.5)

The amplitude-width relation shows that the solitons will exist for the cmKdV equation provided

\[
ab > 0
\]  

(2.6)

Thus, the coefficients of nonlinearity and dispersions both carry the same sign.

2.1 Integrals of motion

The two conserved quantities, or integrals of motion, of cmKdV equation are the energy \( (E) \) and the linear momentum \( (M) \) and they are respectively given by [2]

\[
E = \int_{-\infty}^{\infty} |q|^2 \, dx = 2A \sqrt{6b/a}
\]  

(2.7)

\[
P = i \int_{-\infty}^{\infty} (q^* q_x - q q_x^*) \, dx = \kappa E = 2\kappa A \sqrt{6b/a}
\]  

(2.8)

3 Soliton perturbation theory

The perturbed cmKdV equation that is going to be studied in this paper is given by [2]

\[
q_t + a |q|^2 q_x + bq_{xxx} = i\epsilon R
\]  

(3.1)
where $R$ represents the perturbation terms and $\varepsilon$ is the perturbation parameter. In presence of the perturbation terms denoted by $R$, the adiabatic variation of the soliton energy and momentum are given by [2]

$$\frac{dE}{dt} = \varepsilon \int_{-\infty}^{\infty} (q^* R + q R^*) \, dx$$  \hspace{1cm} (3.2)

$$\frac{dP}{dt} = i\varepsilon \int_{-\infty}^{\infty} (q^* R - q R^*) \, dx$$  \hspace{1cm} (3.3)

These lead to, after simplification, the adiabatic variation of the soliton amplitude and frequency respectively as [2]

$$\frac{dA}{dt} = \frac{i\varepsilon}{2\sqrt{a}} \int_{-\infty}^{\infty} (q^* R - q R^*) \, dx$$  \hspace{1cm} (3.4)

and

$$\frac{dk}{dt} = \frac{\varepsilon}{2A} \sqrt{\frac{a}{6b}} \left[ \int_{-\infty}^{\infty} (q^* R + q R^*) \, dx + i\kappa \int_{-\infty}^{\infty} (q^* R - q R^*) \, dx \right]$$  \hspace{1cm} (3.5)

The slow change of soliton velocity is [2]

$$v = b \left( B^2 - 3\kappa^2 \right) + \frac{i\varepsilon}{E} \int_{-\infty}^{\infty} x(q^* R - q R^*) \, dx$$  \hspace{1cm} (3.6)

where $E$ is the energy of the soliton that is given by (2.7).

### 3.1 Perturbation terms

In this subsection, some particular perturbation terms are going to be taken into consideration [5, 6]. Thus,

$$R = i \left\{ \alpha |q|^4 q_x + \beta \left( |q_x|^2 q_x + \gamma (qq_{xx})_x q^* + \delta q_x q_{xx} q^* + \lambda qq_{xx}^* q_x + \nu q_{xxxx} \right) \right\}$$  \hspace{1cm} (3.7)

Thus, the perturbed cmKdV equation is given by

$$q_t + a |q|^2 q_x + b q_{xxx} = -\varepsilon \left[ \alpha |q|^4 q_x + \beta \left( |q_x|^2 q_x + \gamma (qq_{xx})_x q^* + \delta q_x q_{xx} q^* + \lambda qq_{xx}^* q_x + \nu q_{xxxx} \right) \right]$$  \hspace{1cm} (3.8)

Here, in (3.8), the right hand side represents the higher order corrections of the solitons. The first two terms in the right hand side of (3.8) are considered with full nonlinearity and $n$ is the full nonlinearity factor. These perturbation terms adiabatically deform the soliton amplitude and frequency as

$$\frac{dA}{dt} = 0$$  \hspace{1cm} (3.9)
4 Conclusions

This paper studied the adiabatic dynamics of solitons due to the perturbed cmKdV equation where the perturbation terms are taken with full nonlinearity. This kind of a problem arises in the studies Plasma Physics. By the aid of SPT, it was proved that the amplitude and the width of the soliton stays constant when the perturbation terms are turned on, while the soliton frequency undergoes the adiabatic variation. Thus the dynamical system leads to a fixed point that is stable. Hence the soliton amplitude (and the width) as well as the frequency gets locked to a fixed value.

In future, this analysis will be extended to study the multiple-scale analysis of the perturbed cmKdV equation and thus obtain the quasi-stationary solitons in presence of such perturbation terms with full nonlinearity. The numerical analysis of such solitons will also be studied and these results will be reported for publication in future to other journals. Additionally, the stochastic perturbation terms will also be taken into account.

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References


