Invariant canonical form for the multiple logistic regression

Christos P. Kitsos 1*

1 Department of Mathematics
Technological Educational Institute of Athens
12210 Egaleo, Athens, Greece

* Corresponding Author. E-mail: xkitsos@teiath.gr

Abstract. This paper introduces the invariant canonical form of the Generalized Logistic Regression. Therefore, the appropriate group of (affine) transformations is introduced, while a rival one is also discussed and both are compared. The former transforms the whole model, while the latter generalizes the one introduced by Kitsos (2006), for the transformation of the variables for the simple logistic model.

1 Introduction

Risk Analysis has been a bonafide segment of all decision-making in sciences: Economics, Engineering, Biology etc., and is applied eventually for prediction. In principle, prediction is related to forecast with either interpolation or extrapolation. In Cancer Risk Assessment, there is perhaps the only necessity for backwards extrapolation: "low dose extrapolation".

One of the most appropriate statistical methods to evaluate the Relative Risk (RR) in practice, for example in dose-response experiments involving one or more variables, is the logistic regression. The construction of local $D$- or $c$-optimal designs, for the binary response models have been discussed by Sitter & Torsney (1995) among others, while Ford et al. (1992) and Kitsos (2006) introduced for the one variable logistic model the canonical form.

While to fit a logistic regression model appears an aesthetic appeal, when investigating the Relative Risk, the experimental design problems still is under consideration. Recall that the logistic model is a nonlinear model, although it is intrinsic linear model, and therefore any design depends on the unknown parameters we are planning to estimate as well as possible. Practically, this means that the optimal design points depend on the involved parameters, see Kitsos & Kolovos (2010), who worked on $D$- and $c$-optimality for calibrating pH-meters. In logistic regression, Breslow & Day (1980), Cotti & Rigas (2007) among others, it is important to evaluate the Odds Ratio (OR). For one of the input $X_i$ variables, $i = 1, 2, ..., p$, in the multiple logistic model is associated with the estimate of

2010 Mathematics Subject Classification: 62J12, 62A05, 62K05.
Keywords: Logit model, Invariance, Orbit, Experimental design, Multiple logistic regression.
the parameter $\hat{\beta}_i$. Then, the OR or equivalently the Relative Risk (RR), is estimated by computing the difference $e^{\hat{\beta}_i} - 1$ which estimates the percentage change (increase or decrease) in the odds $\pi = \Pr(Y = 1)/\Pr(Y = 0)$ for every 1 unit in the input variable $X_i$ holding all the other $X_j$’s fixed, $j \neq i$, $i = 1, 2, \ldots, p$. The optimal design approach, with more than one input variables, faces the problem that these variables might be either categorical or continuous.

In the simple logistic regression, there are cases where a second order degree polynomial for the logit transformation is considered. In such cases, the log-odds transformed data forms a second-degree polynomial. We are examining how robust is the model, in grounds of the optimal experimental design theory, Fornius (2008). For the Quadratic Logistic Models she provided, in her PhD thesis, a number of $D$- and $c$-optimal designs, while a number of simulations performed for a sequential -optimal design.

In simple logistic regression, a transformation was adopted: so that to have a "basis", a "reference" to start performing the experiment. This is, roughly, the so-called canonical form of the logit model. The main positive characteristic is that the problem turns to be, due to the transformation, independent of the unknown parameters.

The design space is crucial for obtaining the $c$-optimality, due to Elfving’s theorem, see Pukelsheim (1993) and Kitsos et al. (1988), Zarikas et al (2010) for applications. Thus, the canonical form turns to be essential in such studies, Ford et al. (1992), Kitsos (2006), Fornius (2007). If we simply replace the parameters (acting as coefficients) multiplied by the input variables, with a new variable, this might provide easier calculations but there is not a theoretical justification for it, Haines et al. (2007). We emphasize that for design purposes the input variables need to be continuous. This paper attempts to provide the mathematical background for such cases and extend the canonical form to multiple logistic regressions.

Therefore, in Section 2, the main points of the invariance approach are discussed, in Section 3 the (affine) group of transformations is introduced, and in Section 4 the half-Risk principle is introduced.

2 The Background of Invariance

Let $\{(X, \mathcal{A}, \mathcal{P})| P \in \mathcal{P}\}$ be a family of probability spaces and let $X$ be distributed according to a probability distribution $P_0$, $\Theta \in \Theta$ with $\Theta \subseteq \mathbb{R}^n$ known as parametric space and let $p \geq 1$. Let $g$ be a one-to-one transformation of $X$ onto itself. We denote by $gx$ the image of $x \in X$ under $g$, the collection of all sets of the form $gA$ rises the sigma-field $g\mathcal{A}$, while $g\mathcal{P}$ is the probability measure on $g\mathcal{A}$, in the sense that $g(P(gA)) = P(A)$. Therefore, an isomorphism between $(X, \mathcal{A}, \mathcal{P})$ and $(gX, g\mathcal{A}, g\mathcal{P})$ is created and any function $f$ on $X$ generates $gf$ in the sense $f(x) = gf(gx)$ and $f$ is an $\mathcal{A} = \mathcal{P}$ integrable function on $X$, then $gf$ is a $\mathcal{A} = \mathcal{P}$ integrable function on $gX$.

As it can be easily proved, for given $g$, $g'$ two such transformations, there exist $gg'$ and $g^{-1}$, defined trivially, see Lechman (1997), Giri (1996). Therefore, the set $G$, of all such transformations, defines a group. Two kinds of transformations is of a general interest: the group of location changes, known also as translations, $gx = (x_1, x_2, \ldots, x_n) + c$, $c \in \mathbb{R}$, $x \in \mathbb{R}^n$, and the group of positive scale changes, $gx = c(x_1, x_2, \ldots, x_n)$, $c > 0$, $x \in \mathbb{R}^n$.

Now, for a given $x \in \mathbb{R}^n$ there is an equivalent $x' = gx \in \mathbb{R}^n$ for $g \in G$. Thus, the group $G$ of transformations partitions $\mathbb{R}^n$ into equivalence classes or orbits. Any statistic $t(x)$ is said to be invariant under $G$, when $t(gx) = t(x)$, $x \in \mathbb{R}^n$, $g \in G$, i.e. remains constant on each orbit. Due to the orbit development, the maximal invariant function is defined, as that function which admits different value on each orbit, i.e. it is invariant under $G$, and $t(x) = t(x') \Rightarrow x' = gx$, where $g \in G$ and $x, x' \in \mathbb{R}^n$,
see also Fraser (1979). Ellerton et al. (1986), adopted this approach to choose the optimal order of a response polynomial, while Mueller and Kitsos (2004) evaluated invariant tolerance regions.

The (affine) transformations discussed above (i.e. the translation and scale) were left invariant and give rise to left invariant differentials, similarly we can consider the right transformations, which provide the right differentials, and both can be related through a modular function. For such a development on evaluating particular measures see Mueller and Kitsos (2004) as far as the tolerance regions are concerned.

It was pointed out that invariance provides grounds for applications to a number of statistical methodologies and to more practical problems: In Cryptology, the sense of invariance is very important, Bauer (1994). Indeed, given two texts \( T = (t_i) \) and \( T' = (t'_i) \), \( i = 1, 2, \ldots, k \) then the, so called, kappa function provides an index of coincidence and the chi function is defined, known also as the cross product according to Kullback, need to be invariant. The integer \( k \) is the number of symbols in the alphabet under consideration, i.e. \( k = 26 \) for the English alphabet.

In biological applications if we can transform the generalized logistic model to a simpler form, we need less prior information to perform an experiment. In optimal biological experimental design problems (Bioassays), Kitsos (1999, 2005) the prior information is essential to define the optimal design points. With the logistic regression we evaluate the Relative Risk, see Kitsos (2010) for the Brest Cancer problem. In the next session we are extending the canonical form to the \( p \)-variable Logit model, providing the appropriate group of affine transformations.

### 3 The Canonical Form of the \( p \)-variable Logit Model

We study the case, when performing an experiment, the response, say \( Y \), has two outcomes, usually denoted by 0 or 1. Such a response is known as binary response, and is linked with the explanatory variable \( X = (X_1, X_2, \ldots, X_p) \), through a probabilistic model \( T(x; \theta) \) of the form:

\[
T(x; \theta) = P(Y = 1|x) = p(x) = 1 - P(Y = 0|x),
\]

with \( \theta \) being the vector of involved parameters and \( x \) a value of \( X \). Typical examples in Cancer Bioassays, Kitsos (2005), for \( T(x; \theta) \) are the Logit and Probit models, and the explanatory variable, denoting exposure to risk, to be binary i.e. \( x = 0 \) or 1. For discussion on Cancer Problems, see Breslow and Day (1980). To estimate the probability \( p(x) \), a well known \( 2 \times 2 \) contingency table is obtained, Kotti and Rigas (2005), for the simple Logistic Regression model of the form:

\[
p(x) = \frac{e^{\beta_1 + \beta_2 x}}{1 + e^{\beta_1 + \beta_2 x}} = T(x; \theta), \quad \theta = (\beta_1, \beta_2).
\]

Model (3.2) is within the class of the Cancer multistage models, Kitsos (1999), and for the optimal design of it, see Silvey (1980). When \( p \) input variables \( x = (x_1, x_2, x_3, \ldots, x_p) \) associated with \( \theta = (\beta_1, \beta_2, \ldots, \beta_p) \) parameters are considered, then model (3.2) can be generalized to

\[
p(x) = \frac{\exp(x\theta^t)}{1 + \exp(x\theta^t)},
\]

where \( \theta^t \) represents the transpose of vector \( \theta \).

**Proposition 3.1.** For the logistic model, as in (3.3), and for the Fisher’s \( p \times p \) information matrix \( \mathbf{I}(\theta, \xi) \), there exists a vector \( v \) such that, \( \mathbf{I}(\theta, \xi) = v^tv \), with \( v^t \) the transpose of \( v \).
Proof. If we let \( z = x^t \beta \) then the log-likelihood function is of the form \( \ell = \log \{ T(z)Y - T(z) \} + \text{const} \). Therefore,
\[
I(\theta, \xi) = E \{(\nabla \ell)(\nabla \ell)^t \} = T^t [T(1 - T)] w^2, \quad (3.4)
\]
with \( v = T'[T(1 - T)]^{1/2} u \) and \( v' = T'[T(1 - T)]^{1/2} u' \), and so, the Proposition was proved. \( \square \)

For the simple Logit model (3.2), the "canonical form" has been introduced see, Ford et al. (1992), Kitsos (2006). We extend this idea to \( p \)-variable logistic model. Indeed: for the Logit transformation of (3), we have the logistic regression model
\[
y = \log \frac{p(x)}{1 - p(x)} = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + ... + \beta_p x_p + \sigma e, \ e \sim N(0,1), \quad (3.5)
\]
with \( \sigma > 0 \) being a scale unknown parameter. We would like to have a group of affine transformations to obtain the canonical form for it, generalizing for model (3.3) the canonical form of (3.2). Therefore we state and prove

**Proposition 3.2.** The set of the (affine) transformations \( G \) as

\[
G = \left\{ g = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \beta_1 & \beta_2 & \cdots & \beta_p & \sigma \end{pmatrix} : g \in \mathbb{R}^{(p+1)^2} \right\}, \quad (3.6)
\]

with \( I_p = \text{diag}(1,1,...,1) \in \mathbb{R}^{p \times p}, \ O = (0,...,0) \in \mathbb{R}^{1 \times p}, \ \theta = (\beta_1,\beta_2,...,\beta_p) \in \mathbb{R}^{1 \times p} \) and \( \sigma \in \mathbb{R}^+ \), forms a group, under matrix multiplication.

**Proof.** Considering two elements \( g, g' \in G \), with
\[
g' = \begin{pmatrix} I_p & O' \\ \gamma & \sigma \sigma' \end{pmatrix} \quad \text{and} \quad \theta' = (\beta'_1,\beta'_2,...,\beta'_p) \in \mathbb{R}^{1 \times p}, \ \sigma' \in \mathbb{R}^+.
\]

Then for \( g \) and \( g' \) can be easily verified that
\[
gg' = \begin{pmatrix} I_p & O' \\ \gamma & \sigma \sigma' \end{pmatrix} = h,
\]
with \( \gamma = (\gamma_1,\gamma_2,...,\gamma_p) \in \mathbb{R}^{1 \times p} \) and \( \gamma_r = \beta'_r + \sigma' \beta_r, \ r = 1,...,p \). Therefore, \( h \in G \). The unit element \( id. \) of \( G \) can be defined as
\[
id. = \begin{pmatrix} I_p & O' \\ 1 \end{pmatrix} = \text{diag}(1,1,...,1) \in \mathbb{R}^{(p+1) \times (p+1)}, \quad (3.7)
\]
and trivially \( g \cdot id. = id. \cdot g = g \). For given \( g \) the inverse \( g^{-1} \in G \) can be defined as
\[
g^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ -\sigma^{-1} \beta_1 - \sigma^{-1} \beta_2 \cdots - \sigma^{-1} \beta_p \sigma^{-1} \end{pmatrix} \quad (3.8)
\]
Notice that from the definition of the multiplication
\[ g^{-1}g = gg^{-1} = \left( \begin{array}{c} \mathbb{I}_p \\ \gamma \sigma^{-1} \end{array} \right) \in \mathbb{R}^{(p+1)^2}, \]
with \( \gamma_r = -\sigma^{-1}\beta_r + \sigma^{-1}\beta_r = 0, r = 1, \ldots, p. \) Therefore, \( g^{-1}g = gg^{-1} = id, \) with \( id \) the unit element of the defined group, as in (3.7). Moreover, it can be easily verified that for given elements \( g_1, g_2, g_3 \in G \)
with
\[ g_j = \left( \begin{array}{c} \mathbb{I}_p \\ \gamma_j \sigma_j \end{array} \right) \]
and \( \beta_j = (\beta_{j1}, \beta_{j2}, \ldots, \beta_{jp}), j = 1, 2, 3, \) we have \( g_1(g_2g_3) = (g_1g_2)g_3. \) This is true, as \( \sigma_1(\sigma_2\sigma_3) = (\sigma_1\sigma_2)\sigma_3 \) and \( \beta_{ii} + (\beta_{i1} + \beta_{i2}\sigma_2)\sigma_1 = \beta_{ii} + \beta_{i1}\sigma_1 + \beta_{i2}\sigma_2\sigma_1 = (\beta_{ii} + \beta_{i1}) + (\beta_{i2}\sigma_2). \)

**Corollary 1.** For \( g \in G \) as above, it holds:
\[ \det g = \sigma > 0, \ det(g - \lambda I) = (1 - \lambda)^p(\sigma - \lambda) \] and \( tr g = \sigma + p. \)

With \( \sigma = 1 \) all the eigenvalues of \( g \) are equal to \( \lambda = 1. \)

Now, let the logistic regression (3.5) is of the form
\[ y = \log \frac{p(x)}{1 - p(x)} = \sum_{i=1}^{p} \beta_i x_i + \sigma e = X\beta + \sigma e, \] (3.9)
with \( X = (x_{ij}), i = 2, \ldots, p, j = 1, 2, \ldots, n \) and, in particular, \( x_{1j} = 1, \) i.e. \( x_1 = 1. \) Equivalently, this is written, in the (technical) form
\[
\begin{pmatrix}
  x'_1 \\
  x'_2 \\
  \vdots \\
  x'_p \\
  y
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1 \\
  \beta_1 & \beta_2 & \cdots & \beta_p
\end{pmatrix}
\begin{pmatrix}
  x'_1 \\
  x'_2 \\
  x'_p \\
  e'
\end{pmatrix},
\] (3.10)
or, in matrix notation, \( \psi = gE, \) where \( g \) is an element of the transformation group \( G, \) as in (3.6). For the transformed form (3.10), the orbit of a point \( \psi \) is
\[ G\psi = \{ g \in G | g \in G \} \leftrightarrow G\psi = \left\{ \sum_{i=1}^{p} \beta_i x_i + ce, \ \beta_i \in \mathbb{R}, \ c \in \mathbb{R}^+; i = 1, 2, \ldots, p \right\}. \]

Notice that, for any other model \( z, \) as in (3.9), then for \( Gz \) and \( G\psi \) holds: \( G\psi \cap Gz = \emptyset \) or \( G\psi = Gz. \)

As neither \( \beta_i \geq 0, i = 1, 2, \ldots, p \) nor \( \Sigma_{i=1}^{p} \beta_i = 1, \) the set \( G\psi \) cannot be considered as an affine simplex, but expands a \( p \)-dimensional linear space.

**Theorem 3.3.** Consider any positive definite matrix \( M \) with \( \det M \neq 0 \) and vector \( c \) with the appropriate dimension, so that, \( N = c^TM^{-1}c \) is valid. Then, \( N \) remains invariant if \( c \) and \( M \) transformed under \( g. \)

**Proof.** Let \( c = (c_1, c_2, \ldots, c_p, c_{p+1})^T, \) and \( M \) a \((p+1) \times (p+1)\) matrix. Then,
\[ N = c^TM^{-1}c = c^Tg^T(g^{-1})(M^{-1})(g^{-1})gc = (gc)^Tg^TM^{-1}g(gc) = c_gM_g^{-1}c_g, \]
with \( c_g = (c_1, c_2, \ldots, c_p, c_{p+1})^T, \) \( c_{p+1} = \Sigma_{i=1}^{p} \beta_i c_{p+1} + \sigma e \) and \( M_g = g^TM g, \) in the transformed space. \( \square \)
**Corollary 2.** If $M = M(\theta, \xi)$ is the average per observation information matrix, with $\xi$ being a design measure, then $N = N(\theta, \xi)$ is the transformed form of $M(\theta, \xi)$.

**Corollary 3.** If $p = 2$, then the c-optimal design remain invariant under $g$.

Corollary 2 generates the Elfving’s c-optimality, which holds for being a ”ray”.

**Theorem 3.4.** There is a transformation for $g \in G$ as in (3.6) to $h \in G$ with the vector of parameters $\beta$ equals to $\beta \in (\beta_1, 1, \ldots, 1)$.

**Proof.** Let $g = \begin{pmatrix} I_p & 0 \\ \hat{\theta} & \sigma \end{pmatrix}$ and $\hat{\theta} \in (\beta_1, \beta_2, \ldots, \beta_p)$.

We would like to transform $g$ to $h$ with

$$h = \begin{pmatrix} I_p & 0 \\ \beta & \sigma \end{pmatrix}$$

and $\beta \in (\beta_1, 1, \ldots, 1)$ element of $G$, as in (3.6). Indeed, let $Q$ a $(p+1) \times (p+1)$ matrix such that

$$h = QgQ^{-1},$$

(3.11)

with

$$Q = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix},$$

thus $Q^{-1} = \begin{pmatrix} R^{-1} & 0 \\ 0 & 1 \end{pmatrix}$.

Relation (3.11) is eventually equivalent to $\beta = \theta^{-1}R^{-1}$ which holds with $R^{-1} = \text{diag}(1, \beta_2^{-1}, \ldots, \beta_p^{-1})$, and therefore, $R = \text{diag}(1, \beta_2, \ldots, \beta_p)$.

**Corollary 4.** The canonical form of the multiple logistic model is the transformed (3.3) to

$$p = p(z) = \frac{e^{\beta_1z_1 + \beta_2z_2 + \ldots + \beta_pz_p}}{1 + e^{\beta_1z_1 + \beta_2z_2 + \ldots + \beta_pz_p}} = \frac{e^{\beta z^*}}{1 + e^{\beta z^*}}.$$

(3.12)

**Proof.** Let $\psi = gE$ as in (3.10) and $Q$ as in (3.11). Then,

$$z = Q\psi = QgE = QgQ^{-1}QE = hQE = hE^*,$$

with $h$ as in (3.11) and $E^*$ the transformed confounded with the parameters error vector. Thus, we moved from orbit $G\psi$ to $Gz$, the canonical form, with $p$ as in (3.12).

Model (3.12) is the “canonical form” (the affine transformed model) of the given logistic model, see Figure 1. Indeed: Fig 1(b), where $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 5$, the given logistic model is presented. In Fig. 1(a), with $\beta_1 = 1$, $z_2 = 2x_2$, $z_3 = 5x_3$, the canonical form of the Fig. 1(b) model is presented.

**Example 1.** Haines et al. (2007) considered the logistic dose response model of the form $\text{logit}(p) = \beta_0 + \beta_1d_1 + \beta_2d_2$ with $d_1, d_2 \geq 0$ being doses. They simply replace $\beta_1d_1$ and $\beta_2d_2$ by $z_1$ and $z_2$. Actually, this is in accordance with the statistical framework presented here. This replacement is the (affine) transformation to the canonical form of the logistic model. If we consider transformation (3.11) with $\theta = (\beta_1, 1, 1)$, $z = (1, z_1, z_2)$, following Haines et al (2007), then denoting by
and following Proposition 3.1 \( vv^t \) is the Fisher’s information (transformed matrix) \( M_g = g^t M g \) (see Theorem 3.3) and not the Information matrix for the given model. Moreover, \( \det M_g = (\det g)^2 \det M = \sigma^2 \det M \) and thus, \( M_g \) and \( M \) produce the same \( D \)-optimal design with \( M_g \) being, for the one point design, equal to

\[
M_g(\beta; z) = \exp(\beta z^t) \left[ 1 + \exp(\beta z^t) \right]^2 z z^t.
\]

The discussion above provided the theoretical background for the methods adopted to applications.

4 Half-Risk Procedure

The Relative Risk for the variable \( X_i \) in model (3.3) is \( RR_{X_i} = e^{\hat{\beta}_i} \), while for the transformed \( z_i \) variable is \( RR_{z_i} = e^{\hat{\beta}} \). Therefore, the ratio

\[
\rho = \frac{RR_{X_i}}{RR_{z_i}} = e^{\hat{\beta}_i - 1},
\]

provides the change of moving from the canonical form to the one under investigation. If we want to see the percentage change from \( z \)-space to \( x \)-space then the ratio \( \rho' = (e^{\hat{\beta}_i} - 1)/(e - 1) \), is more appropriate.

The idea of 50\% cut might save money and time to any Risk Analysis problem and therefore, to Biostatistical Analysis. We shall try to see how we can perform the cut-in-half -risk. Consider that the transformed logit model the linearly independent points \( z_i, i = 1, 2, ..., p \) of \( R^n \) is of the form (3.12). If we access weights \( w_1 = \frac{\beta_1}{\beta_1 + (p-1)}, w_i = \frac{1}{\beta_1 + (p-1)}, i = 2, 3, ..., p \), so that \( \sum_{i=1}^{p} w_i = 1 \), then the set

\[
A_s = \left\{ \sum_{i=1}^{p} w_i z_i, \text{ with } \sum_{i=1}^{p} w_i = 1, w_i \geq 0, i = 1, 2, ..., p, z_i = 1, \beta_1 \geq 0 \right\},
\]

is an affine simplex.

**Example 2.** In Example 1, Haines et. al. (2007) assigned (or assumed) a positive weight \( w_i \) to all the transformed variables. This is because an (affine) simplex \( A_s \) asked to be formed, while the design was depending on the constant term, as it should.

Sitter & Torsney (1995) as well as Haines et al. (2007) for the two variable dose-response model (actual for the canonical form of the two qualitative variable logit model) considered as design space the \((z_1, z_2)\) region, and chose to work with a "lane" between two parallel lines and not to coincide with lines of constant logit.

The coefficients \( w_i, i = 1, ..., p \) are the barycentric co-ordinates relative to \( z_i, i = 1, 2, ..., p \). The constant term is associated with the larger weight. The affine simplex is the convex hull of its vertices, and there is a homeomorphic function with any other affine simplex \( A' \). This is equivalent to say that the vertices correspond (1-1 and onto) to each other. Adopting the line of thought of homotopy theory, Hilton (1965) among others, showed the following results:

If \( d \) is the diameter of \( A_s \), then, for every simplex of the normal subdivision \( A'_s \), the diameter \( d' \leq \frac{d p^t}{p+1} \). The \( A'_s \) can be defined as
**Fig. 1** Graphs of (a) $y = \exp(1 + z_2 + z_3)$, and of (b) $y = \exp(1 + 2z_2 + 5z_3)$.

$$A'_s = \left\{ \sum_{i=1}^{p} w_i \left( \frac{p}{p+1} z_i \right), \text{ with } \sum_{i=1}^{p} w_i = 1, w_i \geq 0, i = 1, 2, \ldots, p \right\},$$

and is a simplex with vertices $\frac{p}{p+1} z_i, 0 \leq i \leq p$, see Figure 2. If $p$ is (sufficiently) large then the subdivision of $A_s$ to $A'_s$ can be sequential up to $\nu$-subdivision which has diameter $d_s \leq \left( \frac{n}{p+1} \right)^\nu d$.

**Fig. 2** Diameter and the half-risk evaluation due to it.

This discussion provides an algorithm to construct a portion of Relative Risk (simplex of the) input variables:

I. Consider the $p$-variable model.

II. Define the desirable portion $(\frac{1}{2}, \frac{1}{4}, \ldots)$. We shall recommend choosing half-risk, i.e. $\frac{1}{2}$.

III. Define the diameter and the new portion of variables.

**5 Discussion**

This theoretical result practically means that we can work as follows: perform the experiment with the "easiest" scale and position parameters. Then, transfer the results with an element within the group of affine transformations and still we have an optimal design. Returning to the initial design space, through the inverse transformation, still we are moving within a class of optimal designs.
That is, perform the remaining experiments within the class of designs generated by the group of transformations provided the input variables are not discrete. When we fit the model still the canonical form is appreciated: to see how far we moved from it.

Acknowledgments

I would like to thank Dr. Thomas Toulias, Department of Mathematics, Technological Educational Institute, for his comments and work with the figures. The referee’s comments were very much appreciated and helped the author to provide the final version of the paper.

References