Mathematical model for microanalysis of socio-economic behavior

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Abstract. A novel mathematical model of micro socio-economic behavior is proposed. It is based on the new concept of the informational channel, viewed as an elementary socio-economic cell. The processes, taking place within this cell, are described by a set of partial differential equations that relate the two main aspects of a socio-economic system: energy (any material asset, such as money, material property, etc.) and information.

The solution is obtained in the form of the Volterra integral equation that relates the value of disturbance upon the socio-economical cell and the value of the wealth function, which is defined as an integral characteristic of both material and informational components.

Two basic cases of disturbance were analyzed: a constant and a single-pulse flux. In both cases, the results show a quite natural behavior. The proposed model might serve as a basis for more advanced analysis of socio-economical systems.

1 Introduction

It is well-known that contemporary models for analyzing the socio-economic behavior of markets in response to changes happening in the environments, in which these markets function, still cannot provide accurate predictions of market behavior [Du (2004)]. Recently some attempts to apply fractal
and chaos theories have been made [Chiang (1992), Du (2004), Peters (1991), Peters (1994)]. Another approach is the stochastic way of description of complex socio-economic systems; see, for example, Tsay (2002).

It seems natural to assume that, albeit quite complex, the behavior of socio-economic systems should be fully governed by the flow of material assets and information through and within their boundaries. Postulating a relationship between information and energy [Laszlo (2003)], it seems that it should be possible to model the behavior of socio-economic systems in the same way as the behavior of complex physical systems is modeled, that is, by the use of methods of mathematical physics [Kulish (2004)].

Furthermore, the behavior of socio-economic systems, in the same way as the behavior of other physical systems – whatever the complexity of the system is – should comply with the fundamental principles such as, for instance, energy conservation, the second law of thermodynamics, and the least action principle. Hence, it is proposed that it should be possible, starting from the very fundamental principles, to develop a mathematical model – a set of equations – that will adequately describe the behavior of socio-economic systems [Kulish (2004)].

The classical model of the Brownian motion proposed first by Bachelier (1900) for market behavior modeling and then independently by Einstein in 1905 may well serve as a good example [Einstein (1926)]. A larger particle moves – evidently by random – when placed in an environment where it experiences multiple collisions with the molecules constituting the environment. Yet, the particle’s random walks are described by the diffusion equation, that is, an exact equation that is used to describe energy transport in many physical systems. Moreover, the diffusion equation is directly derived from the fundamental principles mentioned above.

To extend this analogy to the behavior of socio-economic systems is to say that, although not predictable in every detail, this behavior can be described by means of deterministic equations with predictable solutions.

One very important remark is to be made at this point. It is known from chaos theory that solutions of deterministic equations can be chaotic, that is, non-predictable [Glendinning (1995)]. It is often believed that the behavior of socio-economic systems is given by such chaotic solutions and thus cannot be predicted. Yet, the recent discovery of bios and the biotic analysis of market fluctuations show that, although a chaotic part is present in market fluctuations, their major part are biotic, that is, might be well predictable [Sabelli (2005)]. Moreover, it seems that a stable socio-economic system cannot be characterized by a fully chaotic behavior, for such a behavior would very soon lead to a partial or even complete collapse of the system in question.

It is argued in this paper that the major trends in the behavior of socio-economic systems can be described by partial differential equations. Furthermore, it will be shown that these trends are uniquely determined by the form of influences acting upon the system, whereas some properties of the socio-economic system play the role of parameters in the solutions.

Thus, the approach, proposed in this paper, may provide an alternative tool for analyzing the market behavior.

During last decades, there were many attempts to develop an adequate market model based on idea of behavior description by means of partial differential equations [Gale (1973), Medio (1992), Michener and Ravikumar (1998), Medio and Raines (2005)]. Some of they quite successfully describe particular situations within the market, although there still is a lack of a general theory that could be really used in practice. This paper ia an attempt to develop some basic concepts for such a theory. In this paper, the approach is based on the idea that adequate description of the market should include
some model of socio-economic dynamics, in which the evolution of society in the economical aspect will appear as the solution for the set of differential equations.

Of course, the problem in question is quite complex. However, it looks reasonable to develop a simplified model that still would reflect the main features of the process. To develop such a model, it is necessary to firstly choose variables and parameters that are able to adequately describe these main features. In this paper, it is suggested that the socio-economic behavior of an individual within a given socio-economic environment is a function of both energy (material) and informational (non-material) components, where “energy” is meant in the broadest possible sense (e.g., money, material property, etc.).

2 Model formulation

The model is based on the concept of a socio-economical cell (micro-informational channel), which is characterized by three parameters: the “informational distance” between the end-points of the channel, the energy density per this channel, and the wealth function due to this channel. Consequently, a point in the phase space of this model represents an individual equipped with an informational channel that consumes energy and allows the individual to exchange information with others to change the value of his/her wealth function. Thus, the model involves three variables: wealth, $W$, energy density, $\varepsilon$, and the informational distance, $\eta$, which is defined as a measure of closeness between two persons in the informational sense. Hence, $\eta = 0$ corresponds to the informational equivalence (the individual is infinitely close to his/her counterpart at the other end of the informational channel; immediate sharing of all the available information).

The wealth function, $W$, is an integral characteristic of the individual’s material and informational property. Thus, in this model, it is proposed that the person’s wealth depends on both person’s material and informational property that may include person’s abilities and potentials.

It is suggested that the time evolution of a single socio-economic cell is described by a set of the following partial differential equations:

$$\begin{align*}
\frac{\partial \varepsilon}{\partial t} &= D_\varepsilon \nabla^2 \varepsilon - r \varepsilon \eta \\
\frac{\partial W}{\partial t} &= D_W \nabla^2 W + k \nabla W
\end{align*}$$

(2.1)

In equation (2.1), $W = W(x, \eta, t)$, where $x$ is, in general, a multi-dimensional spatial variable that in a certain way describes the set of socio-economical cells. The informational distance $\eta = \eta(x, \varepsilon, t)$, while the energy density per channel $\varepsilon = \varepsilon(x, \eta, t)$. The equations contain the parameters: $D_\varepsilon$, $D_W$, $r$, $k$.

Here, it is assumed that the wealth function of a cell depends on the informational distance in this cell and may vary with time. The reason for this dependence is, in particular, that for a channel that impacts the value of wealth, the impact should grow with the decrease of the informational distance of the channel. The informational distance, in turn, may vary with time and the amount of energy spent for maintaining the channel. On the other hand, the energy itself may depend on the informational distance, because the informational distance plays a role of the phase co-ordinate in this model.

The model describes the time evolution of wealth and the energy content within the socio-economical cell by differential equations. There are some natural ideas behind these equations. Firstly, the time variation is due to the diffusion process that tends to dissipate the quantity between two points with different values. Unlike the classical diffusion models that require the direction of dissipation from a higher potential towards a lower one, this model allows for any direction of diffusion. The
diffusion terms are given by $D_\varepsilon \nabla^2 \varepsilon$ and $D_W \nabla^2 W$, where $D_\varepsilon$ and $D_W$ denote diffusion coefficients for energy and wealth, respectively. Secondly, the time variation is as well due to the presence of sources or sinks within the system. This is modeled by the terms $-r\varepsilon \eta$ in the energy equation and $k\nabla W$ in the equation for wealth (2.1). In case $r > 0$, the term $-r\varepsilon \eta$ models the energy consumed for maintaining the information channel, where $r$ is the consumption rate. In case $r < 0$, the energy (material assets) is produced from the use of the informational channel. The term $k\nabla W$ describes wealth production from the informational channel. It is suggested that this production is proportional to the gradient of wealth with respect to the informational distance. Indeed, in case of a small value of gradient, the amount of wealth produced is not very sensitive to the informational distance. On the other hand, the higher the production of wealth, the stronger it should depend on the informational distance of the channel. The values of $k$ are allowed to be both positive and negative, where $k < 0$ corresponds to an informational channel that increases wealth (a source channel), while $k > 0$ represents the case of a wealth-consuming channel (a sink channel). If $k = 0$, then such a channel neither produces nor consumes wealth and is called “neutral” channel.

Thus, the model depends on four scalar parameters: $D_\varepsilon$, $D_W$, $r$, $k$.

3 Solution procedure

In order to treat Eqn. (2.1), we now apply the technique that was first discussed by Oldham and Spanier [1974]. Whilst the equations can be solved by using Laplace transforms, the technique adopted allows for obtaining integral equations that relate local values of the intensive properties $\varepsilon$ and $W$ and the corresponding local values of their fluxes. The same method was successfully used in numerous applications (Kulish and Lage [2000], Kulish et al. [2001], Kulish and Lage [2002]). In recent works, the method was extended and applied to problems involving combustion (Kulish and Novozhilov [2003(1)]), hyperbolic heat transfer (Kulish and Sourin [2002], [2003]), turbulent flows (Kulish [2004(2)]), and such problems in biomedical engineering as modeling of the neural response to an external stimulus (Kulish and Novozhilov [2004]) and the alveolar gas exchange (Kulish [2004(3)]).

Consider the one-dimensional version of Eqn. (1.1),

$$\frac{\partial W}{\partial t} = D_W \frac{\partial^2 W}{\partial \eta^2} + k \frac{\partial W}{\partial \eta}$$

(3.1)

with the initial condition $W(\eta, 0) = W_0$.

In this study, for the sake of clarity, only positive values of the wealth diffusivity, $D_W > 0$, are considered. The case of the negative $D_W$ will be studies in the following paper.

Upon introducing the excess variable $\hat{W}(\eta, t) = W(\eta, t) - W_0$, equation (3.1) becomes

$$\frac{\partial \hat{W}}{\partial t} = D_W \frac{\partial^2 \hat{W}}{\partial \eta^2} + k \frac{\partial \hat{W}}{\partial \eta}$$

(3.2)

with the initial condition $\hat{W}(\eta, 0) = 0$.

Taking the Laplace transform of (3.2),

$$\frac{d^2 \Phi}{d\eta^2} + \frac{k}{D_W} \frac{d\Phi}{d\eta} \frac{s}{D_W} \Phi = 0$$

(3.3)
where \( \Phi(\eta; s) \) is the Laplace transform of the excess variable \( \hat{W} \) and \( s \) is the Laplace transform variable.

The general solution of Eqn. (3.3) is

\[
\Phi(\eta; s) = C_1(s) \exp \left( -k - \frac{\sqrt{k^2 + 4D_W s}}{2D_W} \eta \right) + C_2(s) \exp \left( -k + \frac{\sqrt{k^2 + 4D_W s}}{2D_W} \eta \right) \tag{3.4}
\]

Note that, for \( D_W > 0, \sqrt{k^2 + 4D_W s} - k > 0 \) and, since the solution must remain bounded as \( \eta \to \infty, C_1(s) = 0 \). Hence, the solution reduces to

\[
\Phi(\eta; s) = C(s) \exp \left( -k + \frac{\sqrt{k^2 + 4D_W s}}{2D_W} \eta \right) \tag{3.5}
\]

Upon differentiating the solution with respect to \( \eta \) and rearranging the terms,

\[
\Phi = -\frac{k}{2\sqrt{D_W}} \frac{d\Phi}{d\eta} \tag{3.6}
\]

Applying the shift and convolution theorems and noticing that the inverse Laplace transform \( \mathcal{L}^{-1} \left[ \frac{1}{\sqrt{\pi t^2}} \right] = \frac{1}{\sqrt{\pi t}} - ae^{zt} \text{erfc}(a\sqrt{t}) \) [Abramowitz & Stegun (1965)], where the complementary error function, \( \text{erfc}(z) \), is defined as \( \text{as} \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt \), the solution becomes

\[
W(\eta, t) = W_0 - \sqrt{D_W} \int_0^t \frac{\partial W(\eta, \tau)}{\partial \eta} \left[ \frac{e^{-k^2/4D_W(t-\tau)}}{\sqrt{\pi(t-\tau)}} - \frac{k}{2\sqrt{\pi}} \text{erfc} \left( \frac{k}{2\sqrt{D_W}} \frac{t-\tau}{D_W} \right) \right] d\tau \tag{3.7}
\]

Upon substituting the relationship between \( W \) and its flux \( \phi \), defined as \( \phi(\eta, t) = -D_W \frac{\partial W}{\partial \eta} \), into (3.7),

\[
W(\eta, t) = W_0 + \frac{1}{\sqrt{D_W}} \int_0^t \phi(\eta, \tau) \left[ \frac{e^{-k^2/4D_W(t-\tau)}}{\sqrt{\pi(t-\tau)}} - \frac{k}{2\sqrt{\pi}} \text{erfc} \left( \frac{k}{2\sqrt{D_W}} \frac{t-\tau}{D_W} \right) \right] d\tau \tag{3.8}
\]

Equation (3.8) provides a relationship between the values of the wealth function \( W \) and the flux of wealth with respect to the informational distance \( \eta \). The equation is valid for any \( \eta \), including \( \eta = 0 \). The latter case corresponds to the situation when the individual’s wealth state undergoes external influences given by the flux function \( \phi(0, t) \).

The case of \( \eta = 0 \) corresponds to an informational channel, along which information can be accessed, shared and transmitted instantaneously and with no limitations (an individual with respect to him/herself or an individual with an instantaneous informational access within a group). For this case, the energy equation in (2.1) reduces to

\[
\frac{\partial \varepsilon}{\partial t} = D_e \nabla^2 \varepsilon \tag{3.9}
\]

This equation can be treated by the same method that has been applied above for solving the wealth equation. The solution is

\[
\varepsilon(0, t) = \varepsilon_0 + \frac{1}{\sqrt{4\pi Dt}} \int_0^t \frac{\psi(0, \tau)}{\sqrt{(t-\tau)}} d\tau \tag{3.10}
\]
where $\varepsilon_0$ is the energy density in the equilibrium state ($t = 0$) and $\psi(\eta, t)$ is the energy flux with respect to $\eta$, defined as $\psi(\eta, t) = -D_\varepsilon \frac{\partial \varepsilon}{\partial \eta}$.

In case $\eta \neq 0$, the energy equation still can be solved analytically, but the solution is not so simple and involves the inverse Laplace transform of the Airy functions, $Ai(s)$. This case is beyond the scope of this paper and will be considered in future works.

4 Model validation

In this section, the model is validated in its limit case $\eta = 0$.

For a given constant value of the diffusion coefficient $D_W$ ($D_W = 1$ in this study), numerical integration of equation (3.8) is straightforward, and is carried out by explicit time advancement using trapezoidal rule. In order to handle singularity, analytical integration is performed in the vicinity of the upper limit. The time step is chosen in such a way that the relative error of the solution does not exceed $10^{-6}$. Upon achieving this criterion, solution becomes independent of further reduction in time step.

Figure 1 shows how the evolution of the wealth function depends on the value of the parameter $k$ in case of a constant flux of wealth, $\varphi = 1$. It can be easily seen from the figure that when $k = 0$, the wealth function grows with time as $\sqrt{t}$. It is well-known that such behavior appears in the Brownian motion and diffusion processes. When $k$ is negative, however, the informational channel serves as a source of wealth, and the growth of the wealth function well exceeds the classical diffusion. On the contrary, when $k$ assumes a positive value, the informational channel works as a sink of wealth; the growth of the wealth function is dumped and becomes much slower than the classical diffusion. Therefore, in this case, wealth does not practically increase, although a constant positive flux of wealth is present.

Thus, the parameter $-k$ can be viewed as a measure of the usefulness of the informational channel in the sense of the wealth acquisition, and the case $k = 0$ can be termed as a “neutral” – nether source nor sink – channel.

![Fig. 1](image-url)  
**Fig. 1** Time evolution of the wealth function in case of a constant unit flux of wealth (different $k$).
Figure 2 shows how the evolution of the wealth function depends on the value of the parameter $k$ in case of a unit Gaussian pulse of the wealth flux, which is modeled as

$$
\phi(\eta, t) = \exp \left[ -\left( \frac{t - \mu}{\sigma} \right)^2 \right]
$$

(4.1)

where the temporal variable $t$ is again measured in arbitrary units. For the sake of illustration, the flux given by (4.1) with the mean $\mu = 1$ and the variance $\sigma = 0.5$ was used.

On Fig. 2, the value of the wealth function is normalized as $w = (W - W_0)/(W_{\text{max}} - W_0)$, where $W_0$ is an arbitrary equilibrium (initial) value, whereas $W_{\text{max}}$ is the maximal value of wealth acquired in case $k = -1$.

Hence, the figure shows how the memory of the system about a single unit pulse of wealth varies with time. This again depends on the value of the parameter $k$ – the usefulness of the informational channel with respect to the wealth acquisition. When the value of $k$ is negative (a source channel), the memory decays slowly, so that the impact of the single unit pulse of wealth is well-remembered long after the pulse itself has been ceased. On the other hand, when the value of $k$ is positive (a sink channel), the memory decays very fast, so that not only the impact of the pulse becomes fast forgotten, but also the value of wealth may become eventually zero. The case $k = 0$ corresponds to the memory of a neutral channel.

To illustrate how the wealth function depends on the diffusion coefficient, the value of $k$ was fixed ($k = -1$) and Eqn. (3.8) was solved numerically for three different values of the diffusion coefficient ($D_W = 1.0; 5.0; 10.0$), both in the case of the constant flux ($\varphi = 1.0$) and in the case of a single pulse modeled by Eqn. (4.1).

Figure 3 shows the results in the case of the constant flux.
Figure 4 shows the results in the case of a single pulse modeled by Eqn. (4.1). On Fig. 4, the value of the wealth function is normalized as \( w = (W - W_0)/(W_{\text{max}} - W_0) \), where \( W_0 \) is an arbitrary equilibrium (initial) value, whereas \( W_{\text{max}} \) is the maximal value of wealth acquired in case \( D_W = 1 \).

The solution of the energy equation in case \( \eta = 0 \) (Eqn. 3.10) behaves in the same way as the solution for the wealth function in case \( k = 0 \) (see Figs. 1 and 2). This solution describes energy consumption for maintaining the individual’s existence when this individual’s informational channel is “frictionless”. In such a case, for example, it follows from the obtained solution that the growth of the wealth function implies a corresponding growth of energy consumption, which looks quite natural – “the more wealth the more consumption”.

Fig. 3  Time evolution of the wealth function in case of a constant unit flux of wealth (different \( D_W \)).

Fig. 4  Time evolution of the wealth function in case of a unit Gaussian pulse of the wealth flux (different \( D_W \)).
5 Conclusions

In this paper, a novel mathematical model of micro socio-economic behavior is proposed. The model is based on the concept of the informational channel, viewed as an elementary socio-economic cell. The processes, taking place within this cell, are described by a set of partial differential equations that relate the two main aspects of a socio-economic system: energy (any material asset, such as money, material property, etc.) and information (any non-material asset, such as knowledge, skill, potential, contacts, etc.).

The solution is obtained in the form of the Volterra integral equation that relates the value of disturbance acting upon the socio-economical cell (flux) and the value of the wealth function, which is defined as an integral characteristics of both material (energy) and informational components.

In order to perform initial validation of the model, two basic cases of disturbance (flux) were analyzed: a constant flux and a single pulse of flux. In both cases, the results show the behavior that is quite natural from the socio-economical point of view. Thus, the proposed model might serve as a basis for more advanced analysis of socio-economical systems.

References


