Computational model of Fermionic Society: Social polarization study via dynamical mean-field theory

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Abstract. We propose a computational model of ‘Fermionic Society’. It is based on the fermionic Hubbard model, which is used for computational study of the Socioeconomic Polarization. For this purpose, a dynamical mean-field theory of the Hubbard model has been implemented in C#. The simulation has been performed for a range of values of the socioeconomic repulsion potential, which forms the polarization vector-field. A brief review of the range of methods used for social science simulation shows other methods currently commonly in use. Newer methods based on or inspired by quantum physics are also briefly reviewed. A Socio-Cognitive sciences perspective is presented to ground the current work in its sociological roots, leading to a discussion on Socioeconomic Polarization with respect to an archetypal socioeconomic problem to illustrate how polarization and depolarization applies to concrete problems.

1 Introduction

In this paper we propose a computational model of ‘Fermionic Society’, based on the fermionic Hubbard model [1]. The main order-parameter of the proposed Fermionic Society model is the Socioeconomic Polarization, which is itself modeled upon the the physical phenomenon of ferromagnetic polarization, or magnetization, a spatiotemporal vector-field that represents the density of magnetic dipole-moments in ferromagnetic materials. The key is that temperature can be considered as the control parameter. While the magnetization is random at some temperature, as the temperature is varied over its full range, then at some particular value the magnetization becomes strictly polarized, where the magnetization takes either one value or another, spin-up or spin-down, magnetic-north-pole or magnetic-south-pole. If the temperature is varied back over that threshold then the magnetization again becomes random.

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Recently, a number of research projects have been increasingly addressing the issues of social polarization within the developed economies (see e.g. Berg and Ostry [2] and Moulaert [3]; also note earlier work by Okun [4]). From our particular organisational perspective the importance of understanding economic influences in the study of armed conflict in the contested urban environment has been recently emphasized; see [5, 6]. In fact it is Kilcullen’s [5] seminal observation, that the central novel feature of this modern conflict is the highly networked nature of the urban world, which encompasses complex flow systems of economics trade, people, idea-systems, information with competition for not only territory, consisting of the natural land and built environs, but the built environment services such as, water, electricity, transportation, communication and the like. The economic competition does not, however, stop there; it is also a competition for idea-systems in what some have called the ‘battle for ideas’ [7].

The simulation has been performed in C# for a range of values of the socioeconomic repulsion potential, which forms the polarization of a vector-field. The implementation is based on QUEST (see [8]).

1.1 Methods in Social Science Simulation

Gilbert and Troitzch [9] review methods typically used in simulation for the social sciences, and a summary diagram is shown in Figure 1. The grey area, shows continuous equation based models, while the white area shows object, event or agent based models.
It should be noted that there are no quantum based models here in Figure 1. The left hand side of Figure 1, represents a Systems Dynamics approach is the closest to it, but quantum based models are effectively a new stream of simulation methods stemming from the node of “Differential Equations” of 1700, some of which are reported in [10]. There these new quantum based computational devices, such as the Recurrent-Quantum-Neural-Network (RQNN), shown in Figure 2, initially proposed by Behera [11, 12, 13] are discussed in [10]. It is important to understand that we call them computational devices because they can be applied as models of a wide range of problems. For instance Behera [11] used it as a model of the saccades of eye movements, while Ivancevic et al [14, 15, 16, 17] used it to model the turbulent movement of individual agents in civilian crowd scenarios.

**Fig. 2** Diagram of a Recurrent-Quantum-Neural-Network (adapted and modified from [11, 12, 13]).

### 1.2 Early techniques used to study social segregation

From a simulation perspective Schelling [18] was among the first to study social segregation using cellular automata as the implementation vehicle. Whilst in principle from a mathematical perspective cellular automata are computationally quite powerful, in practice the implementation typically used in simulation for the social sciences are quite limited (see [18] and [19, 9]). A summary on Schelling’s migration model and others, may also be found in Gilbert and Troitzch [19, 9].

### 2 A Socio-Cognitive Sciences Perspective

Before proceeding it necessary to understand the frame of reference of sociologists, social-psychologists (sociology), social-psychologists(psychology) and finally psychologists\(^1\). For just like other sciences

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1. While this four groups are obviously related to some extent, it is interesting to note that they publish in separate journals, and it would appear that they generally do not talk to each other that much. For instance introductory books on social-
there can be at times specific terms that have a precise technical meaning, which does not necessarily concur with the common English usage of the the terms. For the purposes of this paper we will take largely a sociological stance as we are mainly concerned with modelling societal level phenomena. Important aspects of a sociological perspective of society will be covered in Section 2.1.

For reasons of computational efficiency it would be convenient to be able to treat at times social groups of individuals as effectively as a single agent. The question arises, however, is there any justification from a socio-cognitive perspective from taking such a point of view? By examining the concepts of social identity, social groups and social emotions it will be shown that there is a plausible defence for taking such a stance.

2.1 The seven aspects of society

Sociological theory according to Macionis and Gerber [22] consists of seven components. They are: culture, society, socialization, social interaction in everyday life, groups and organizations, sexuality and society and finally deviance.

With respect to culture it is necessary to understand it in terms of a way of life and the elements of culture per se. The Way of life is a construct that what people do and what people have, which has two aspects, the material, meaning physical things or material symbols, and the non-material, meaning non-material symbols, such as ideas, language, values, beliefs and norms, which are the elements of culture. Symbols refers to anything carrying meaning, while language is considered as a system of symbol manipulation and of course, for interpersonal communication. Values are the defined standards for the how good something is, and beliefs are statements or assumptions that people hold to be true, while norms are the rules of expected behavior. Norms consists of mores, which are widely observed standards in the society and are of held to be of great moral significance and folkways, which by contrast are standards for routine casual interaction.

A society then is defined as, people who interact in a defined territory and share a culture. Traditionally, of course a territory refers to a geographic region which has defined physical boundaries. For example this society exists in the territory which lies south of the river and the mountains, while that other society is in the territory that lies between the area north of the river and the ocean. In the social sciences it is an aphorism say that physical boundaries create social boundaries. We will return to the concept of social boundaries in a subsequent section. In a more modern interpretation, however, it is possible to begin to consider territory as some space in some virtual world, say on the internet. For example, my territory is a island in the computer game Second-life, or perhaps these two forums are my territory, while those other 500 are not.

Within a society the process of socialization refers to life long learning through social experience that contributes to the development of individual personality and allows full participation in that particular society. Social interaction in everyday life refers to the “rules” that guide behavior in everyday situations. Social interaction is the process by which people act and react in relation to others. These everyday interactions include both emotions and language.

Groups and organizations are important sociological terms to understand. A social group has two or more people who interact and identify with one. The group size may small, e.g. a sports team or an infantry section, or large, e.g. Port Adelaide Football Club supporters, or an infantry brigade. The group will act according to some sort of general rules and the social group may be a primary or secondary group. A primary group will be a smaller personal group, e.g. those mates who are Sturt...
Sexuality is constructed by society and is a theme found in most areas of society, and while it is biological, society, including patterns of culture and inequality, shapes how sexuality is experienced. It does not just shape a person’s thoughts about themselves, but also creates/shapes the impressions of others about that person. Social issues and controversies involving sexuality and society include teen pregnancy, abortion, prostitution, pornography, date rape, and sexual violence amongst many others.

Deviance refers to the state of standing out by not conforming to what is expected behavior or a norm and is a reflection of both individual choice and the norms and the conventional operation of a particular society. It is the recognisee violation of social rules and cultural norms. Norms guide almost all human activities, ranging from minor infractions, such as bad manners, to major ones, like serious violence.

There are three theories of deviance; biological that says deviance is instinctual or innate in the individual, psychological, which purports that deviance is due an unsuccessful socialization of the individual concerned and conformational that views norms, and deviance from them, as constructs of the society itself. The problem with the first two theories is that they provide only a limited understanding of crime and other deviance because most violations are done by normal people. The third, however, recognizes that what is deviant varies from place to place and over the history of time, according to the current prevailing cultural norms. So particular behaviors and certain individuals become deviant as either other individuals, or society as a whole, construct, or if you like define, them to be so. The final point to make about this third theory of deviance is that it is also a reflection of who does or does not have social power.

2.2 Social identity

Turner [23], following earlier work of Tajfel [24, 25], developed the concept of self-categorization, which can be use to determine whether the individual perceive themselves as a personal identity or a social identity. When interacting with in-group members, the individual’s personal identity dominates, and they think of themselves in terms of their specific individual abilities, goals, personality traits, physical attributes, and so on. In contrast when the individual is interacting with other out-group members, now the social identity has come to be important. They think of themselves as a group member and will view themselves, on the whole, in terms of archetypal group attributes and characteristics.

2.3 Social emotions for social groups

Social groups, following Macionis and Gerber [22], have already been defined above in Section 2.1, as was social identity in Section 2.2. We can now turn to the subject of social emotions, where there are at least two points of view as to what they are. Burnett et al [26] take the view that social emotions require the representation of the mental state of “the other” within the particular individual concerned, while Smith et al [27, 28, 29] takes the view that a social emotion is felt by the particular individual
when their social identity is salient. In this latter interpretation, events happening to the group are not felt on behalf of the group, but as part of the extended self. Effectively the boundary of self has moved to include the group [27], extending the appraisal theoretic point of view (see [30, 31, 32, 33, 34, 35, 36, 37, 38, 39]) for the individual whilst being in a personal identity mode. Smith was interested in developing a new conceptualization of prejudice, and defined prejudice as a “social emotion experienced with respect to one’s social identity as a group member, with an out-group as a target”.

For the purposes of this paper, we take Smith’s perspective, as it gives a justification for being able treat agents, some of which are social groups and other of which are individuals, as a cohesive whole. If nothing else this allows us to achieve computational efficiency, but we also believe it allows us to propose models which can be flexibly focused on the key aspects of the particular concrete problem at hand. In other words, relatively unimportant aspects\(^2\), can be modeled summarily, while the salient aspects can be modeled in more detail.

### 2.4 Social boundaries

It is possible to discuss boundaries in many contexts from personal boundaries to the Berlin Wall; e.g. see[40]. With respect to the social sciences it is an aphorism to say that “physical boundaries create social boundaries” [22]; this is illustrated, for instance, by Allaire [41] with respect to neighbor boundaries in the context of town planning. For example, two adjacent neighborhoods separated by a river, may over time develop quite different social norms [41].

On the other hand cultural differences can influence how we design a build physical boundaries. Ozaki et al [42] consider this, for example, in the context of the meaning of social space and illustrate this by studying how a Japanese house, as opposed to a British one, is designed and then constructed.

It is also possible to consider social boundaries that are not based on physical boundaries. For instance, social boundaries drawn on the basis of socioeconomic parameters, like income, housing-standards, and healthcare-standards, may create just as real effects on social inequality, as a physical boundary, such as the Berlin wall, and this will be examined in Section 2.5.

### Lewin’s Life-space

Lewin is regarded as the “Father of social-psychology(psychology)” and had a strong interest in topological and vector psychology [43, 44]. This enables a wide range of psycho-physical problems and interactions to be modeled. For instance, the interactions that were of interest to [45] were between an embodied agent and firstly, the physical world, secondly, his individual psychological world and finally the in-groups and out-groups of his social world, and between these three broad groups it is trivially obvious that the system is fundamentally highly nonlinear.

A summary of Lewin’s behavioral approach appears in [10], while a briefer version from [45] is repeated here. Lewin’s most famous equation (first mentioned in [46]) can be formulated as a disjoint union of the following functional mappings (indicated by ‘→’, for \(i = 1, 2, \ldots; a = 1, 2, \ldots; \alpha = 1, 2, \ldots\)):

\[
\bigcup_{i,a,\alpha} \{(P_a, E_i) \rightarrow B_\alpha\} \quad \Rightarrow \quad B = f(P, E) \quad \Leftrightarrow \quad B_\alpha = f(P_a, E_i) \tag{2.1}
\]

where \(B = \{B_\alpha\}\) denotes the total behavior composed from the set of the modes of behavior \(B_\alpha\); \(P = \{P_a\}\) is the person’s total mental state consisting of the set of various mental states and structures

\(^2\) However, but not so irrelevant that they may be omitted altogether.
$P_a; E = \{E_i\}$ is the psycho-biological environment with differing environmental regions $E_i$. In toto, $f$ is a function of $P; E$, the essential components of the total situation. The first two terms in Lewin’s equation are his own formulation, while the last term is a modern tensorial one. The environment he referred to was a psycho-biological-space and to distinguish it from the physical space, he called it life-space [46].

Lewin was inspired by field-theoretic interpretations of classical and relativistic physics, with their Euclidean and Riemannian geometries, which he applied in his field–theory, comprising of what he called topological and vector psychology, [43]. The function $f$ in (2.1) is quite general and abstract and may be formulated in a number of ways to suit the problem at hand (see e.g. [45]).

**Generalization of life-space to a social-group-space**

Although Lewin’s life-space is intended to model the psycho-biological-space of an individual, if that individual is in a mode where social identity is salient, then it is easy enough to see how it can be extended to a social-group-space. Further, [47] indicates how in principle Lewinian principles may be applied to problems concerning social groups and societies, based on earlier work reported in [48]. This is somewhat similar to the way Smith [27] extended the appraisal theoretic point of view of [32, 34, 38] into social emotion mentioned earlier in Section 2.3.

**Field Theory Constructs**

In brief [45, 10] summarizes the nine constructs the field–theory of [49] as follows:- 1. **Position** for location and the related geometry of relative position (i.e., topology); 2. **Locomotion** or relation of geometric position over time; 3. **Cognitive structure** for the relative position of various geometrical objects (points, boundaries, regions, etc.); 4. **Force** for the tendency to locomotion and change over time, and the accompanying force-field for the spatial distribution of forces; 5. **Goal** for a force-field focused on a particular point, with a positive valence; aversion is similar but with a negative valence; 6. **Conflict** for at least two overlapping competing force-fields, which may result in a vacillatory equilibrium for certain constellations of overlapping force-fields; 7. **Fear** which is related to the psychological-future; 8. **Power** which refers to a “possibility of inducing forces” on another person via a power-field, e.g., a parent’s power over a child; 9. **Values** refers to internal or self regulating power, which Lewin in comparison to Freud’s psychoanalytical theory thought could be the internalized-parent [49, pp.197-198]. In this paper we will only be implicitly concerned with the constructs of cognitive structure, specifically social boundaries, and power, specifically social-power. Both social boundaries and social power are points of emphasis that were first mentioned in Section 2.1.

**2.5 Social inequality and Economic inequality**

Following the approach of Smith et al [27, 28, 29] social inequality can arise when some specific social out-group is prejudiced against or victimized, by another individual, social group or groups, or even by the rest of the society as a whole, who hold, for some reason or other, social-power over the out-group. For example, in the context of crime and violence, see Cohen [50] and Morenoff [51]. Social inequality leads at some level to social segregation, as shown by Schelling [18]. On the other hand economic inequality based, say, on say household income, will not necessarily lead to social inequality, in practice, it usually does (see Berg and Ostry[2] and Okun [4]).

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3 This refers to psychological conflict within the life-space, not violence in a physical space.
2.6 Social and Socioeconomic Polarization

Social polarization typically indicates that the system response is a bi-modal phenomena with respect to the system input parameters. For instance, instead of a set of student scores for some examination results being represented, as they normally are, by some unimodal distribution, typically a Gaussian or bell-shaped distribution, there are two polarized extremes, for some set of parameters. One of those modes might be such that, say, the upper mode might represent the distribution of scores for some relatively advantaged socioeconomic group, the “rich-kids”, while the, say, the lower mode may represent the distribution of scores for some relatively disadvantaged group, the kids in extreme, dire and desperate “poverty”.

Apouey [52] reports on such socioeconomic polarization effects, based on age and income for healthcare outcomes for French women, Wessel [53], O’Loughlin [54], Pahal [55] and Moulaert et al [3] report for wider society wide welfare issues. For the sake of this introductional report we will focus on the Apouey [52] problem, where the inputs are age and income and the output is an excellence of health parameter, as an archetypal concrete example of socioeconomic polarization. Although Apouey was also concerned with measures of socioeconomic polarization we will only focus on the results of the data, which were the health outcomes for French women. Apouey created six age groups: 18-24, 24-34, 35-44, 45-54, 55-64 and 65+. For the lower two age group there is a strong and clear polarization effect, while there is some ambiguity mid-range for 35-44, their are still two distinct peaks at 55-64. This is then followed by a depolarizing effect in the 65+ group where one of the peaks severely diminishes. This is exactly analogous to the physical ferromagnetic example described in Section 1, except here the system control variable is age, not temperature.

Although Apouey did not indicate casual reasons as to why the system responds this way, we can postulate a possible set of reasons, for the sake of making the story complete, even if it from now on in, it is a complete fiction. The story might then run like this. In their youth, there are two distinct possible health outcomes strictly following their socioeconomic status. Those from poorer socioeconomic backgrounds, having had on the whole parental guidance that was wanting, tend to make bad lifestyle choices resulting in lower standards of health outcomes. As age increases, the picture becomes fuzzier for both poles, since those who started well may now be making poorer choices, and conversely those who started poorly may have learnt from their experience. Finally in old age the situation depolarizes completely, for no matter where you start, old age eventually equalizes everything.

So much for the story, the only question that remains is whether or not we can create a computational device capable of representing such a polarized and depolarized scenario? We note that the polarization or depolarization in such a device should ideally only depend on some independent control variable, such as temperature, age or anything. As we will show in subsequent sections, we believe that the proposed device, which in the first instance will be used to create a model of fermionic society, meets this challenge.

3 The Fermionic Society Model

The fermion Hubbard model is a dynamical lattice representation of fermion-to-fermion interactions and the induced inter-fermionic correlations (see Appendix for the basic formalism; for detailed instructions and computer simulations, see [8]; in particular for Quantum Monte Carlo approach, see
The model is formally defined by the **Hubbard Hamiltonian**, which usually includes the following three basic terms: (i) the kinetic energy $H_1$ of the ‘hopping’ fermions, (ii) the interaction energy $H_2$ of a local repulsion among the fermions, and (iii) the chemical potential $H_\mu$ that controls the filling of the fermionic lattice. The standard complete Hubbard Hamiltonian, representing a general (anisotropic) fermionic lattice, is given by (generalized from eq. (1.57) in [58]; see also [59, 60, 61]):

$$H = H_1 + H_2 + H_\mu$$

where we have defined:

1. The kinetic energy term $H_1$ includes the (non-commutative) second-quantization operators of fermionic creation $c_{k\sigma}^\dagger$ and anihilation $c_{j\sigma}$, which respectively describe the creation $c_{k\sigma}$ of a fermion of spin $\sigma$ (in the so-called Wannier state (with the corresponding Bloch state)$^4$) on the lattice-site $k$ and its destruction $c_{j\sigma}$ on the lattice-site $j$; the tensor $t_{kj}$ gives the Hopping energy per sites $kj$; the symbol $\langle kj \rangle$ stands for nearest-neighbor links on the lattice (i.e., fermion hopping is allowed only between adjacent sites).

2. The interaction energy term $H_2$ goes through all $k$ sites and adds an energy $U_k$ (a local Coulomb-type repulsion) if the site is doubly occupied; the term $n_{k\sigma} = c_{k\sigma}^\dagger c_{k\sigma}$ represents the occupation number operator (see Appendix, subsection 5.2).

3. The chemical potential term $H_\mu$ controls the filling of the lattice; in particular, for $\mu_k = 0$ the system is at half-filling (the occupation $\rho = \langle n \rangle = 1$) due to particle-hole symmetry.

In a (somewhat artificial) case of an isotropic lattice (homogenous in both $k$- and $j$-directions), the Hamiltonian (3.1) simplifies into the **grand-canonical Hamiltonian** (see [57]):

$$H = -t \sum_{\langle kj \rangle \sigma} \left( c_{k\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{k\sigma} \right) + U \sum_k n_{k\uparrow} n_{k\downarrow} - \mu \sum_k (n_{k\uparrow} + n_{k\downarrow}).$$

In the case of half-filling ($\mu = 0$), the Hubbard Hamiltonian (3.2) reduces to:

$$H = -t \sum_{\langle kj \rangle \sigma} \left( c_{k\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{k\sigma} \right) + U \sum_k n_{k\downarrow} n_{k\downarrow}.$$  

$^4$Wannier functions constitute a complete set of orthogonal functions (used in solid-state physics). In particular, if the Bloch states (of a single band in a perfect crystal) are given by:

$$\psi_k(r) = e^{ik \cdot r} u_k(r),$$

then the corresponding Wannier functions are:

$$\phi_R(r) = \frac{1}{\sqrt{N}} \sum_k e^{-i R \cdot k} \psi_k(r),$$

$^5$Besides, the fermionic lattice is at half-filling when $\mu = U/2$; as half-filling is often studied, it is convenient to rewrite the Hamiltonian (3.2) as [56]:

$$H = -t \sum_{\langle kj \rangle \sigma} \left( c_{k\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{k\sigma} \right) + U \sum_k (n_{k\uparrow} - 1/2)(n_{k\downarrow} - 1/2) - \mu \sum_k (n_{k\uparrow} + n_{k\downarrow}),$$

which corresponds to a shift in the chemical potential $\mu$ by $U/2$; when this is done, half-filling (conveniently) always occurs at $\mu = 0$ for any value of $t, U$ and temperature $T$. 
In particular, for a 1D chain with \( M \) sites and periodic boundary conditions, Hamiltonian (3.3) is given by (see [62]):

\[
H = -t \sum_{k=1}^{M} \sum_{\sigma} \left( c_{k+1,\sigma}^\dagger c_{k,\sigma}^\vphantom{\dagger} + c_{k,\sigma}^\dagger c_{k+1,\sigma}^\vphantom{\dagger} \right) + U \sum_{k=1}^{M} n_{k\uparrow} n_{k\downarrow},
\]

(3.4)

where the periodic boundary conditions are expressed by \( c_{M+1,\sigma}^\dagger = c_{1,\sigma}^\dagger \), and \( c_{M+1,\sigma} = c_{1,\sigma} \). Here, \( c_{k,\sigma}^\dagger c_{k,\sigma} \equiv n_{k\sigma} = 1 \), if an electron with spin \( \sigma \) is located at the site \( k \), otherwise \( n_{k\sigma} = 0 \).

There is also a so-called velocity operator associated with the periodic Hamiltonian (3.4), given by:

\[
v = -i t \sum_{k=1}^{M} \sum_{\sigma} \left( c_{k+1,\sigma}^\dagger c_{k,\sigma} - c_{k,\sigma}^\dagger c_{k+1,\sigma} \right) = -i t J,
\]

where

\[
J = \sum_{k=1}^{M} \sum_{\sigma} \left( c_{k,\sigma}^\dagger c_{k+1,\sigma} - c_{k+1,\sigma}^\dagger c_{k,\sigma} \right)
\]

is called the current operator.

### 3.1 Two limits of the Hubbard society model

**A single site limit: \( t = 0 \)**

In the simplest case, if we set \( t = 0 \) in the Hubbard Hamiltonian (3.2), we get a collection of independent sites of the form (see [8]):

\[
H = U \sum_{k} n_{k\uparrow} n_{k\downarrow} - \mu \sum_{k} (n_{k\uparrow} + n_{k\downarrow}).
\]

(3.5)

The single site model (3.5) is easily solved as follows. In total, we have the following four possibilities: the site being empty \( |0\rangle \), or having a single electron (either spin up \( |\uparrow\rangle \) or spin down \( |\downarrow\rangle \)), or being doubly occupied \( |\uparrow\downarrow\rangle \). These four states are eigenstates of the Hamiltonian (3.5) with eigenvalues 0, \(-\mu\), \(-\mu\) and \(U - 2\mu\), respectively. The main descriptors are: the partition function \( Z \), the occupation \( \rho \) and the energy \( E \). They are respectively given by:

\[
Z = \sum_{a} \langle a | e^{-\beta H} | a \rangle = 1 + 2 e^{\beta \mu} + e^{\beta(2\mu-U)},
\]

\[
\rho = \langle n \rangle = 2 \left[ 1 + 2 e^{\beta \mu} + e^{\beta(2\mu-U)} \right]^{-1} \left[ e^{\beta \mu} + e^{\beta(2\mu-U)} \right],
\]

\[
E = \langle H + \mu n \rangle = Z^{-1} \sum_{a} \langle a | H e^{-\beta H} | a \rangle
\]

\[
= \left[ 1 + 2 e^{\beta \mu} + e^{\beta(2\mu-U)} \right]^{-1} U e^{\beta(2\mu-U)} = \mu \rho - \partial \ln Z / \partial \beta.
\]

A fundamental physical quantity in the Hubbard model (3.2), is the local moment, formally defined by:

\[
\langle m^2 \rangle = \langle (n_{k\uparrow} - n_{k\downarrow})^2 \rangle.
\]

Intuitively, the local moment is zero if the site is either empty (\( |0\rangle \)) or has two oppositely pointed spins (\( |\uparrow\downarrow\rangle \)), and it is 1 if the site has a single electron (spin up \( |\uparrow\rangle \) or down \( |\downarrow\rangle \)).

The local moment is related to the double occupancy: \( d = \langle n_{k\uparrow} n_{k\downarrow} \rangle \), by
\[ \langle m^2 \rangle = \langle n_{k\uparrow} - n_{k\downarrow} \rangle - 2d. \]

In particular, at half-filling (\( \langle n \rangle = 1 \)), the relation between the local moment and the double occupancy becomes: \( \langle m^2 \rangle = 1 - 2d. \)

The Hubbard Hamiltonian (3.2) commutes with the operators:

\[ N_{\uparrow} = \sum_k n_{k\uparrow} \quad \text{and} \quad N_{\downarrow} = \sum_k n_{k\downarrow}. \]

In finding the eigenstates of the Hubbard model, we can consider different sectors of \( N_{\uparrow} \) and \( N_{\downarrow} \) separately (e.g., we can consider a particular electron sector in which \( N_{\uparrow} = 1 \) and \( N_{\downarrow} = 0 \)).

**Limit of no interactions: \( U = 0 \)**

In the absence of interactions (\( U = 0 \)), the Hubbard Hamiltonian (3.2) becomes (see [8, 65, 66]):

\[ H = -t \sum_{kj} c_{k\sigma}^\dagger c_{j\sigma} - \mu \sum_k (n_{k\uparrow} + n_{k\downarrow}). \]

It is only when the interactions are turned off (\( U = 0 \)) that the single-particle sector gives us full information about the model for any particle number.\(^7\) It is not necessary that the hopping \( t \) between different sites be the same for all pairs of sites, or that it be limited to near neighbors, or that the chemical potential be the same on all sites. All that matters is that the Hamiltonian \( H \) be a quadratic form in the fermion creation and destruction operators. In general, to solve any non-interactive Hubbard Hamiltonian which takes the form: \( H = \sum_{kj} h_{kj} c_{k\sigma}^\dagger c_{j\sigma} \), where \( h_{kj} \) is a symmetric real matrix, we simply diagonalize \( h_{kj} \) and allow the resulting energy levels to be filled in a way which satisfies the Pauli exclusion principle.\(^8\)

For \( U = 0 \), we have the following special cases:

1. The 1D Hubbard model on a 1D chain:

\[ H = \sum_{k\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} \quad \text{(where} \quad \varepsilon_k = -2t \cos k). \]

\(^6\) To show this, we begin by considering the commutator of the kinetic energy on a single ‘link’ of the lattice connecting sites \( k \) and \( j \) with the total number of electrons on those two sites, i.e., by computing:

\[ \left[ c_{k\sigma} c_{j\sigma}^\dagger + c_{j\sigma} c_{k\sigma}^\dagger - n_{k\sigma} - n_{k\sigma} \right]. \]

This linear algebra is based on the general rule for the eigenvalues of an \( N \times N \) tridiagonal matrix with ‘\( a \)’ along the diagonal and ‘\( b \)’ above and below the diagonal, with periodic boundary conditions, is

\[ \lambda_n = a + 2b \cos k_n \]

where \( k_n = 2\pi n/N \) and \( n \in Z \) (e.g., we can write down the 8×8 matrix for the Hubbard Hamiltonian (3.2), and determine its eigenvalues).

\(^7\) Besides, for \( U = 0 \), even if one has many particles, they just occupy the single particle states in accordance with the Pauli principle. However, when \( U \neq 0 \), the one particle sector eigenenergies do not tell us much about the energies of the sectors with more particles.

\(^8\) Note here the analogy with normal modes in classical mechanics, where the goal is similarly to eliminate a set of degrees of freedom which couple to each other in favor of ones which do not.
2. The 2D Hubbard model on a square lattice:

\[ H = \sum_{k\sigma}(\epsilon_k - \mu)c_{k\sigma}^\dagger c_{k\sigma} \quad \text{[where} \quad \epsilon_k = -2t \cos k_x \cos k_y \text{]} \; \]

Now, using the Fermi function: \( f_k = \left[ 1 + e^{\beta(\epsilon_k - \mu)} \right]^{-1} \), we can define the partition function \( Z \), per-site average occupation \( \rho \), and average energy \( E \), of the non-interactive Hubbard model: \( H = \sum_{k\sigma}(\epsilon_k - \mu)c_{k\sigma}^\dagger c_{k\sigma} \), as:

\[
Z = \prod_k \left[ 1 + e^{-\beta(\epsilon_k - \mu)} \right], \quad \rho = \sum_k \left[ 1 + e^{\beta(\epsilon_k - \mu)} \right]^{-1},
\]

\[
E = \sum_k \epsilon_k \left[ 1 + e^{\beta(\epsilon_k - \mu)} \right]^{-1}.
\]

Here, the partition function \( Z \) is the generalization of the simple fermion oscillator: \( H = \epsilon c_k^\dagger c_k \) to many independent modes; the average occupation \( \rho \) represents the sum of the occupations of different pieces, while the average energy \( E \) is the sum of the energies of different pieces.

The allowed \( k \) values, together with the dispersion \( \epsilon_k \)-relation, determine the discrete density of states \( N(E) \) which counts the number of ways in which the system can have a given energy \( E \). Formally, \( N(E) \) is defined (via Dirac’s delta-function) by:

\[
N(E) = N^{-1} \sum_k \delta(E - \epsilon_k).
\]  

(3.6)

3.2 Social Polarization:

Dynamical Mean-Field Theory and Computational Analysis in C#

In this subsection, we will study social polarization via dynamical mean-field theory (see, e.g. [59, 65, 61, 57]) of the fermionic Hubbard model, implemented in C#.

A Hamiltonian operator that is quadratic in \( c_{k\sigma}^\dagger \) and \( c_{k\sigma} \) (\( H = \sum_{kj} c_k^\dagger h_{kj} c_j \)) can be solved by diagonalizing the matrix \( h_{ij} \). Dynamical mean-field theory is a method which produces such a quadratic Hamiltonian from the Hubbard model which has quartic terms involving four fermion creation and destruction operators: \( U c_{k\sigma}^\dagger c_{i\sigma}^\dagger c_{i\sigma} c_{k\sigma} \).

Briefly, the mean-field approach starts by expressing the number operators \((n_{k\uparrow} \text{ and } n_{k\downarrow})\) as an average value plus a deviation from the average (see [64, 61, 8]):

\[
n_{k\uparrow} = \langle n_{k\uparrow} \rangle + \langle n_{k\uparrow} - \langle n_{k\uparrow} \rangle \rangle, \quad \langle n_{k\downarrow} \rangle = \langle n_{k\downarrow} \rangle + \langle n_{k\downarrow} - \langle n_{k\downarrow} \rangle \rangle.
\]  

(3.7)

Next, if we substitute (3.7) into the Hubbard interaction term: \( n_{k\uparrow} n_{k\downarrow} \) and drop the ‘small’ product of two deviations: \( \langle n_{k\uparrow} - \langle n_{k\uparrow} \rangle \rangle \langle n_{k\downarrow} - \langle n_{k\downarrow} \rangle \rangle \), we get:

\[
n_{k\uparrow} n_{k\downarrow} \approx n_{k\uparrow} \langle n_{k\downarrow} \rangle + n_{k\downarrow} \langle n_{k\uparrow} \rangle - \langle n_{k\downarrow} \rangle \langle n_{k\uparrow} \rangle.
\]  

(3.8)

\(^9\) Similarly, there is a 3D Hubbard model on a cubic lattice (which is not applicable for society modelling):

\[ H = \sum_{k\sigma}(\epsilon_k - \mu)c_{k\sigma}^\dagger c_{k\sigma} \quad \text{[where} \quad \epsilon_k = -2t \cos k_x \cos k_y \cos k_z \text{]} \; \]
This result can be interpreted as follows: the up spin electrons interact with the average density of
down spin electrons, and similarly the down spin electrons interact with the average density of up
spin electrons; these two terms overcount the original single interaction term, so the product of the
average densities is subtracted off.

Using the mean-field replacement of the form (3.7)-(3.8), the Hubbard Hamiltonian becomes
quadratic. Specifically, in 1D it takes the form inherited from (3.4):

\[ H = -t \sum_{k, \sigma} \left( c_{k+1, \sigma}^\dagger c_{k, \sigma} + c_{k, \sigma}^\dagger c_{k+1, \sigma} \right) + n_{k \uparrow} \langle n_{k \downarrow} \rangle + n_{k \downarrow} \langle n_{k \uparrow} \rangle - \langle n_{k \downarrow} \rangle \langle n_{k \downarrow} \rangle. \] (3.9)

Since \( H \) in (3.9) is quadratic, its solution can be obtained by the diagonalization of an appropriate
matrix. In particular, in the case of ferromagnetism (see, e.g. [63]), we have:

\[ \langle n_{k \uparrow} \rangle = n + m, \quad \langle n_{k \downarrow} \rangle = n - m. \] (3.10)

We want to calculate the energy \( E \) for fixed \( n \) as a function of \( m \) and see whether the minimum is
at \( m = 0 \) (paramagnetic state, no ferromagnetism) or \( m \neq 0 \) (ferromagnetism). As, at least in 1D, the
expectation values \( \langle n_{k \uparrow} \rangle \) and \( \langle n_{k \downarrow} \rangle \) have such a simple (site-independent) form, the up and down
energy levels (\( \varepsilon_{\uparrow k} \) and \( \varepsilon_{\downarrow k} \)) can be written as:

\[ \varepsilon_{\uparrow k} = U(n - m) - 2t \cos k, \quad \varepsilon_{\downarrow k} = U(n + m) - 2t \cos k. \] (3.11)

The above relations (3.7)-(3.11) can be used to formulate a simple and efficient algorithm for the
mean-field ferromagnetism calculations in 1D, as follows:

1. Declare the following data:
   (i) lattice size \( N \) (integer, e.g., \( N = 256 \)),
   (ii) total particle number \( N_{\text{tot}} \) (integer, e.g., \( N_{\text{tot}} = 128 \)),
   (iii) hopping \( t \) (real, e.g., \( t = 1.0 \)), and
   (iv) on-site repulsion \( U \) (real, e.g., \( U = 4.0 \)).

2. Outer loop over all \( N_{\uparrow} = 0, 1, ..., N_{\text{tot}} \);
   (i) set \( N_{\downarrow} = N_{\text{tot}} - N_{\uparrow} \); and
   (ii) define densities: \( n_{\uparrow} = N_{\uparrow} / N \) and \( n_{\downarrow} = N_{\downarrow} / N \).

3. Inner loop over \( N \) allowed momentum values:
   \( k = 2\pi/N \{ -N/2 + 1, -N/2 + 2, ..., N/2 \} \); inside the loop:
   fill up the lowest \( N_{\uparrow} \) and \( N_{\downarrow} \) of the energy levels using (3.11).

4. Within the outer loop:
   (i) add the associated energy values \( E \) to an accumulator \( E_t \) which stores the total energy, normalize
   it to the number of sites (i.e., divide by \( N \)) and add the term \(-U \langle n_{\uparrow} \rangle \langle n_{\downarrow} \rangle \) (which gives the energy for \( N_{\uparrow} \) and \( N_{\downarrow} \));
   and (ii) calculate magnetization \( M_g \) as: \( M_g = (n_{\uparrow} - n_{\downarrow})/(n_{\uparrow} + n_{\downarrow}) \).

5. Repeat the calculation for different \( U \) and \( N_{\text{tot}} \) to get the phase diagram: \( M_g \) vs \( E_t \).

The above algorithm has been implemented in C# language, in the program/simulator HubbardSoc.cs:

```csharp

namespace HubbardSoc { public partial class Form1 : Form { public Form1() { InitializeComponent(); } }
```
The phase transition of the socioeconomic polarization

The above C# code has been simulated for the following 8 values of the socioeconomic repulsion potential: $U = [3.8, 3.9, 4.0, 4.1, 4.2, 4.3, 4.4, 4.5]$ and the phase diagrams have been calculated, relating: social polarization (i.e., ‘magnetization’ - ordinate) vs. social ‘configuration’ energy (abscissa). The wave-like, first-order, phase transition is presented in Figure 3.

In other words, the socioeconomic polarization as a function of the social ‘configuration’ energy undergoes the first-order phase transition ...

For simplicity reasons, we make here a tactical assumption that our fermionic Hubbard society model, in the classical mean-field limit, reduces to the simple $ND$ Ising chain model (see, e.g. a
Wikipedia article), in which for any given spin configuration $\alpha$ [i.e., a set of binary spin-states $\sigma_i = \sigma_i(t)$ which can be either spin-up $\sigma = |↑\rangle$, or spin down $\sigma = |↓\rangle$], the energy $E_\alpha = E_\alpha(t)$ \footnote{The internal energy of the Ising system (3.12) is the average value (over a system in thermal equilibrium) of the energy $\langle E_\alpha \rangle$.} and magnetization $M_\alpha = M_\alpha(t)$ are given respectively by:

$$
E_\alpha = -\sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}, \quad M_\alpha = \sum_{i=1}^{N} \sigma_i. \quad (3.12)
$$

As an additional descriptor, the partition function $Z$ of the Ising chain (3.12) depends also on the temperature $T$ and Boltzmann’s constant $k_B$, and is explicitly given by the weighted sum-over-all-spin-configurations $\alpha$ (a discretized ‘path integral’ over all possible spin configurations):

$$
Z(T) = \sum_\alpha \exp \left( \frac{-E_\alpha}{k_B T} \right),
$$

while the corresponding canonical-ensemble PDF is given by the Boltzmann distribution:

$$
P(E_\alpha, T) = \frac{\exp \left( \frac{-E_\alpha}{k_B T} \right)}{Z(T)}.
$$

Recalling from Section 2.5, Apouey \cite{52} problem, where the inputs are age and income and the output is a excellence of health parameter, as an archetypal concrete example of socioeconomic polarization. For the lower age groups there is a strong and clear polarization effect, and is then followed
by a depolarizing effect in the 65+ group where one of the peaks severely diminishes, which is ex-
acty analogous to the physical ferromagnetic example described in Section 1, except here the system
control variable is age, not temperature. This corresponds to the lower half of the Figure 3 moving
from left to right. Magnetization corresponds here to health-outcomes and the energy to income.

4 Conclusion

In this paper we have proposed a computational model of ‘Fermionic Society’, based on the fermionic
Hubbard model. We have used the dynamical mean-field analysis of the Hubbard model, implemented
in the C# programming language, for computational study of the Socioeconomic Polarization. The
simulation has been performed for a range of values of the socioeconomic repulsion potential, which
forms the polarization vector-field. A Socio-Cognitive sciences perspective is presented to ground the
current work in its sociological roots, leading to a discussion on Socioeconomic Polarization with
respect to an archetypal socioeconomic problem to illustrate how polarization and depolarization
applies to concrete social problems.

5 Appendix

Here we give a brief on the second-quantization formalism that is used in the paper.

5.1 Bosonic second-quantization formalism

Recall that the quantum harmonic oscillator is defined by the Hamiltonian operator:
\[ \hat{H} = \left( \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2} \right) / 2, \]  
(5.1)
where \( m \) and \( \omega \) are oscillator’s mass and frequency, while \( \hat{x} \) and \( \hat{p} = -i\partial / \partial x \) (assuming: \( \hbar = 1 \)) are
the position and momentum operators, which satisfy the canonical commutation relation:
\[ [\hat{x}, \hat{p}] = i. \]  
(5.2)

In addition, the second-quantization treatment of the harmonic oscillator (5.1) requires also the (mutually ‘dual’) creation and annihilation operators, \( \hat{a} \) and \( \hat{a}^\dagger \), respectively defined by:
\[ \hat{a} = (m\omega/2)^{1/2} \hat{x} + i (m\omega/2)^{-1/2} \hat{p}, \quad \hat{a}^\dagger = (m\omega/2)^{1/2} \hat{x} - i (m\omega/2)^{-1/2} \hat{p}, \]
and obeying the commutation relation:
\[ [\hat{a}, \hat{a}^\dagger] = 1, \]
which follows directly from (5.2). Besides, the product of the creation and annihilation operators
defines the number operator: \( \hat{n} = \hat{a}^\dagger \hat{a} \). Using the second-quantization operators \( \hat{a}^\dagger, \hat{a} \) and \( \hat{n} \), the Hamiltonian (5.1) becomes:

\[ \text{[11] Recall that the commutator } [\hat{A}, \hat{B}] \text{ of two quantum-mechanical operators } \hat{A} \text{ and } \hat{B} \text{ is defined as: } [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \text{ Note that an analogous relation in classical (Hamiltonian) mechanics is obtained by replacing the commutator with the Poisson bracket } [A, B]_{PB} \text{ multiplied by } i\hbar; \text{ e.g., in case of position } x \text{ and momentum } p, \text{ their Poisson bracket is } [x, p]_{PB} = 1. \text{ This correspondence is the basis of Dirac’s quantization, also called the canonical quantization: } [A, B]_{PB} = i\hbar [\hat{A}, \hat{B}]_. \]
\[ H = \omega (\hat{a}^\dagger \hat{a} + 1/2) = \omega (\hat{n} + 1/2). \]  
(5.3)

An obvious particular case of (5.3) is a simplified oscillator with the Hamiltonian:

\[ \hat{H} = \omega \hat{a}^\dagger \hat{a} = \omega \hat{n}. \]  
(5.4)

The ground state \( |0\rangle \) of the oscillator (5.1)-(5.3) has the following properties:

\[ \hat{a} |0\rangle = 0, \quad \hat{H} |0\rangle = \frac{\omega}{2} |0\rangle. \]

For example, in the position representation (i.e., in the basis of eigenstates of the \( \hat{p} \)-operator), the ground state \( |0\rangle \),

with components: \( \psi_0(x) = \langle x|0\rangle = e^{m^2 x^2/2} \), satisfies: \( \hat{a} |0\rangle = 0. \)

The excited states of the oscillator are built-up by applying the creation operator \( \hat{a}^\dagger \) repeatedly to the ground state \( |0\rangle \):

\[ \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \]

where \( |n\rangle \) denotes the vector of occupation number states and obeys the relation:

\[ \hat{H}|n\rangle = \omega (\hat{n} + 1/2)|n\rangle. \]  
(5.5)

To express the standard partition function for an arbitrary quantum-mechanical system (see, e.g. [56]):

\[ Z = \text{Tr}[\exp(-\beta \hat{H})], \quad \text{(with } \beta = 1/t) \]

(where \( t \) is the temperature, while the common trace operation \( \text{Tr}[\cdot] \) is the sum of the diagonal elements), one chooses any complete set of states \( |a\rangle \) so that: \( Z = \text{tr}(e^{-\beta \hat{H}}) = \sum_a \langle a| e^{-\beta \hat{H}} |a\rangle \).

It can be shown (by using (5.5) and \( |n\rangle \) as the complete set of states) that the partition function of the harmonic oscillator (5.1)-(5.3) is:

\[ Z = \sum_n \langle n| \exp[-\beta \omega (\hat{n} + 1/2)]|n\rangle = e^{-\beta \omega/2}(1 - e^{-\beta \omega})^{-1}. \]

In particular, for the simplified oscillator (5.4), the partition function is:

\[ Z = (1 - e^{-\beta \omega})^{-1}. \]

Given the partition function \( Z \), the (finite-temperature) expectation value \( \langle \hat{A} \rangle \) of an arbitrary quantum-mechanical operator \( \hat{A} \) is given by:

\[ \langle \hat{A} \rangle = Z^{-1} \text{Tr}[\hat{A} \exp(-\beta \hat{H})]. \]  
(5.6)

In case of the oscillator (5.1), (5.1), the relation (5.6) reduces to the expectation value of the number operator:

\[ \langle \hat{n} \rangle = 1/ \left( e^{-\beta \omega} - 1 \right), \]

which is the Bose-Einstein distribution function.\(^{12}\)

\(^{12}\) Recall that the spin of a particle is its intrinsic angular momentum. The spin-statistics theorem classifies particles according to their spin into one of the two quantum statistics they obey:

(i) **bosons** are particles (such as four force-carrying gauge bosons of the Standard Model) with integer spin (1,2,3, etc.) and symmetric states (or, wave functions), obeying the Bose-Einstein distribution;

(ii) **fermions** are particles (such as electrons, quarks and leptons, which are the key building blocks of matter) with half-integer spin (1/2, 3/2, 5/2, etc.) and antisymmetric states, obeying the Fermi-Dirac distribution.
5.2 Fermionic second-quantization formalism

The Hubbard model is usually written in terms of fermion creation and annihilation operators,\textsuperscript{13} which differ in several aspects from the bosonic operators $\hat{a}^\dagger$ and $\hat{a}$. Firstly, the fermion operators in the Hubbard model are not introduced in terms of position and momentum operators, but they are rather independent operators. Secondly, instead of just one creation and one annihilation operator, in the Hubbard model there is a set of such operators, which are usually distinguished by indices $k,j$ (that label the spatial lattice sites) and $\sigma$ (that labels the electron spin, up $|\uparrow\rangle$ or down $|\downarrow\rangle$), so that one usually writes $\hat{c}_{kj\sigma}$ for the creation operator and $\hat{c}_{kj\sigma}$ for the annihilation operator. As a consequence, the occupation number states are no longer characterized by a single number $n$ (as for a single harmonic oscillator) but instead by a collection of occupation numbers $n_{k\sigma}$, with the states written by the ket: $|n_{k\sigma} n_{j\sigma} n_{j\sigma}\ldots\rangle$.

To be able to describe fermions, these operators, by definition, obey the following \textit{anticommutation relations}:\textsuperscript{14}

\[
\{\hat{c}_{kj\sigma}, \hat{c}_{kj'\sigma'}^\dagger\} = \delta_{kj}\delta_{\sigma\sigma'}, \quad \{\hat{c}_{kj\sigma}^\dagger, \hat{c}_{kj'\sigma'}\} = 0, \quad \{\hat{c}_{kj\sigma}, \hat{c}_{kj'\sigma'}\} = 0.
\]  

(5.7)

An immediate consequence of (5.7) is the \textit{Pauli exclusion principle},\textsuperscript{15} which states that the maximum occupation number of a particular site with a given spin is 1.\textsuperscript{16}

\textit{Notational change:} from now on (as we are consistently using the second-quantization formalism) for the sake of simplicity, we will drop all the ‘hats’. So, we define $c_{j\sigma}^\dagger$ to be the operator which creates an electron of spin $\sigma$ on a lattice site $j$; similarly, $c_{j\sigma}$ is the destruction operator, and $n_{j\sigma} = c_{j\sigma}^\dagger c_{j\sigma}$ is the occupation number operator.

As the most trivial 1DOF example, analogous to the \textit{boson Hamiltonian} (5.4), there is the following \textit{fermion Hamiltonian}:

\[
H = \varepsilon c_{j\sigma}^\dagger c_{j\sigma}, \quad \text{with partition function:} \quad Z = \left(1 + e^{-\beta \varepsilon}\right),
\]  

(5.8)

that describes a simplified fermion oscillator with frequency $\varepsilon$, which represents a single fermion site without spin. Its basis states are: $\{|k\rangle\} = \{|0\rangle, |1\rangle\}$, in which $|0\rangle$ represents an unoccupied site, while $|1\rangle = c_{j\sigma}^\dagger |0\rangle$ is an occupied site. The corresponding (stationary) Schrödinger equation is: $H |k\rangle = E_k |k\rangle$, with $E_0 = 0$ and $E_1 = \varepsilon$. Its Hamiltonian matrix is given by:

\[
\{ H_{jk} \} = \begin{pmatrix}
0 & 0 \\
0 & \varepsilon 
\end{pmatrix}, \quad \text{where} \quad H_{jk} = \langle k | H | j \rangle.
\]

\textsuperscript{13} as opposed to \textit{boson creation and annihilation operators}, which we used so far in the second-quantization treatment of the harmonic oscillator (5.1), (5.1)

\textsuperscript{14} Recall that the \textit{anticommutator} $\{\hat{A}, \hat{B}\}$ of two fermion operators $\hat{A}$ and $\hat{B}$ is defined as: $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. Note here the use of the curly brackets for anticommutators, to distinguish them from the boson commutators $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

\textsuperscript{15} The Pauli exclusion principle governs the behavior of all fermions, while bosons are not subject to it. A more rigorous statement states that the total wave function for two identical fermions is antisymmetric with respect to exchange of the particles, which means that the wave function changes its sign if the space and spin coordinates of any two particles are interchanged.

\textsuperscript{16} Because of the anticommutation relation (5.7) one needs to specify a convention for the relation between a state like $|10100\ldots\rangle$ and the \textit{vacuum state} $|\text{vac}\rangle = |000000\ldots\rangle$. The following two possibilities:

\[
|10100\ldots\rangle = c_{j\sigma}^\dagger c_{j\sigma}^\dagger |\text{vac}\rangle \quad \text{and} \quad |10100\ldots\rangle = c_{j\sigma}^\dagger c_{j\sigma} |\text{vac}\rangle
\]

– differ by a sign, yet they are both correct (provided the chosen one is used consistently).
The 1DOF system (5.8) can be easily generalized to following (no-spin, no-hopping, no-interaction) model with $M$ sites:

$$H = \sum_{k=1}^{M} \varepsilon_k c_k^\dagger c_k, \quad (5.9)$$

which has the basis \{\ket{0}_k, \ket{1}_k\} for each $k$-th site, which all comprize the many-particle basis of $2^M$ states:

$$\{\ket{0}_1, \ket{1}_1\} \otimes \{\ket{0}_2, \ket{1}_2\} \otimes \{\ket{0}_3, \ket{1}_3\} \otimes \cdots \otimes \{\ket{0}_M, \ket{1}_M\} = \ket{n_1, n_2, n_3, \ldots, n_M}.\$$

We label basis states by occupation numbers:

$$\hat{n}_k \ket{l} \equiv \hat{n}_k \ket{n_1, n_2, n_3, \ldots, n_M} = n_k \ket{l}.\$$

The Hamiltonian (5.9) conserves separately the occupation of each site:

$$H \ket{l} = \sum_{k=1}^{M} \varepsilon_k \hat{n}_k \ket{l}, \quad \text{such that } [H, \hat{n}_k] = 0.\$$

$n_k$ is a good quantum number: the basis states \ket{l} are eigenstates, that is, Hamiltonian matrix is again diagonal in this basis. Many-particle energy follows trivially:

$$E_l = \bra{l} H \ket{l} = \bra{n_1, n_2, n_3, \ldots, n_M} \sum_{k=1}^{M} \varepsilon_k \hat{n}_k \ket{n_1, n_2, n_3, \ldots, n_M} = \sum_{k=1}^{M} \varepsilon_k n_k.\$$

For more technical details on the fermionic operators, see [8].

References


