Reliability estimation for Burr type-XII distribution under type-I progressive hybrid censoring scheme

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Abstract. In this paper, we consider the estimation of parameters and reliability function of two parameter Burr type-XII distribution under type-I progressive hybrid censoring scheme. We obtain maximum likelihood estimates of both the parameters and the reliability function. The Bayes estimates of the same parametric functions are also evaluated using Lindley’s approximation under symmetric(squared error) and asymmetric(LINEX) loss functions. We give some numerical illustrations based on simulation study with varying sample sizes as well as censoring schemes. A real data set is also analyzed to show the practical applications of the study.

1 Introduction

The random variable $X$, representing the lifetime of a unit is said to follow Burr type-XII (Burr XII) distribution if its probability density function (pdf) is given by

$$f_X(x;c,k) = ckx^{c-1}(1+x^c)^{-(k+1)}, \quad x > 0, c > 0, k > 0,$$

where $c$ and $k$ are shape parameters. The reliability function of a unit, with pdf (1.1) as its lifetime distribution, for a mission time $t_o$, is given by

$$\bar{F}(t_o) = P(X > t_o) = (1+t_o^c)^{-k}.\quad (1.2)$$

Burr XII is a unimodal distribution and has a non monotone hazard rate, which can accommodate many shapes. [1] , [2] , [3] and [4] have shown that, if the parameter are chosen appropriately, the Burr XII distribution contains the shape characteristics of the normal, log normal, gamma and logistic distributions. Other particular cases of Burr XII distribution include inverted beta, Lomax, Pareto and the log-logistic distribution.

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Keywords: Type-I Progressive hybrid censoring scheme, Maximum Likelihood Estimator, Bayes Estimates, LINEX loss function, Lindley’s approximation.
Statistical inference for both the parameters \( c \) and \( k \) of Burr XII distribution using complete and censored samples has been investigated by many authors. [5] obtained Bayes estimates for both parameters as well as reliability function under the squared error loss function (SELF). [6] derived Bayes estimators of the unknown parameters and the reliability function for this model with failure censored sample. [7] considered the estimation of parameter under type-II censored sample. They obtained the maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE), Bayes and empirical Bayes estimators of \( k \). [8] obtained Bayes estimators of \( k \) and the reliability function, under various loss functions, when parameter \( c \) is known. [9] considered ML estimation of parameters of this distribution using randomly right censored data. [10] provided Bayes prediction bounds for future observations under progressive type-II censoring scheme (type-II PCS). [11] derived ML and Bayes estimators of lifetime parameters of the Burr XII model under type-II PCS while [12] studied the estimation problems for this distribution based on type-II PCS with random removals.

In type-II PCS, the time of termination of the experiment is not under control which may delay the result of the experiment. Moreover, the duration of the lifetest is always associated with cost of the lifetest. Keeping these facts in view, we consider here a censoring scheme called type-I progressive hybrid censoring scheme (type-I PHCS). This censoring scheme facilitates experimenter the flexibility of termination of experiment at a prefixed time as well as withdrawal of units during the test. This scheme was proposed by [13]. They obtained ML and Bayes estimators of the parameter of exponential distribution under this censoring scheme. [14] provides the detailed description of several variants of hybrid censoring along with some important result and their applications. [15] presented the estimation procedures for Maxwell distribution under type-I PHCS.

Type-I PHCS is described as follows. Suppose we place \( n \) units on test in a life testing experiment. The integer \( m \) such that \( 1 \leq m \leq n \), numbers \( R_1, R_2, \ldots, R_m \) and time \( t \) are fixed at the start of the experiment. Let \( X_{1:m:n} \) denotes the time of \( t^{\text{th}} \) failure. After observing \( X_{1:m:n} \), we get \((n-1)\) units remaining. From these \((n-1)\) units we remove \( R_1 \) units randomly. Similarly, at the time of second failure \( X_{2:m:n} \), we remove \( R_2 \) from remaining \((n-R_1-2)\) units randomly, and so on. This process continues till the termination of experiment which occurs at \( T = \min \{X_{m:n:m}, t\} \). When \( X_{m:n:m} > t \) we do not get all \( m \) observations. Let \( d < m \), denotes the number of failures occurred up to time \( t \). In this case we observe the data \( X_{1:m:n}, X_{2:m:n}, \ldots, X_{d:m:n} \) and terminate the experiment by withdrawing \( R_{d}^* = (n - d - \sum_{i=1}^{d} R_i) \) units at time \( t \). When \( X_{m:n:m} < t \), we observe the data \( X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:n:m} \) and terminate the experiment at the time of \( m^{\text{th}} \) failure, that occurred at \( X_{m:n:m} \), by withdrawing \( R_m \) units.

This paper deals with the development of ML and Bayesian estimation procedures for the parameters and reliability function of Burr type XII distribution under type-I PHCS. Rest of the paper is organized as follows. In Section 2, we derive MLEs of both the parameters and reliability function. In Section 3, we use Lindley’s approximation to evaluate Bayes estimates of the said parametric functions under SELF while Section 4 deals with the estimation of same under LINEX loss function. In Section 5, we carry out simulation study and in Section 6, we present the analysis of a real data set and conclude the findings.

### 2 Maximum Likelihood Estimation

Suppose \( n \) identical units are put to test, where the life time of each unit follows Burr XII distribution presented by pdf (1.1). The lifetimes experiment is conducted under type I PHCS described in Section-1. After the termination of the test, we get one of the following two types of sample observations.
We notice that \( x_{d;m:n} \) is such that \( x_{d;m:n} < t < x_{d+1;m:n} < \ldots \ldots < x_{n;m:n} \) and \( 1 \leq d \leq m - 1 \). For simplicity throughout the remaining past of the paper we will use \( x \) to denote \( x_{i;m:n} \). The likelihood function of observed type-I PHCS data for both the cases is given as follows.

\[
L(c,k|x) = \begin{cases} 
A_d \prod_{i=1}^{d} f(x_i) \{ F(x_i) \}^{R_i} \{ F(t) \}^{R_d} & \text{if } x_m > t \\
A_m \prod_{i=1}^{m} f(x_i) \{ F(x_i) \}^{R_i} & \text{if } x_m \leq t,
\end{cases}
\]

(2.2)

where, \( A_d = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\ldots(n - d + 1 - \sum_{i=1}^{d-1} R_i) \) and \( A_m \) can be obtained from \( A_d \) when \( d = m \). Using (1.1) and (1.2), we obtain from (2.2) that

\[
L(c,k|x) = \begin{cases} 
A_d(c^d) \prod_{i=1}^{d} x_i^{c-1} (1 + x_i^c)^{-\{k(R_i+1)+1\}} (1+t^c)^{-kR_d^*} & \text{if } x_m > t \\
A_m(c^m) \prod_{i=1}^{m} x_i^{c-1} (1 + x_i^c)^{-\{k(R_i+1)+1\}} & \text{if } x_m \leq t.
\end{cases}
\]

(2.3)

To evaluate ML estimate for \( c \) and \( k \), we take the first derivatives of logarithm of (2.3) with respect to \( c \) and \( k \). On differentiation of log likelihood, the likelihood equations for \( c \) and \( k \) for Case I are given, respectively, by

\[
\frac{d}{c} + \sum_{i=1}^{d} \log x_i - \sum_{i=1}^{d} \frac{k(R_i+1)+1}{(1+x_i^c)} x_i^c \log(x_i) - \frac{kR_d^* t^c \log(t)}{(1+t^c)} = 0
\]

and

\[
\frac{d}{k} - \sum_{i=1}^{d} (R_i+1) \log(1+x_i^c) - R_d^* \log(1+t^c) = 0.
\]

(2.4)

(2.5)

From (2.4) and (2.5), we observe that, it is not possible to get explicit solutions of these equations. Therefore, we evaluate the values of \( c \) and \( k \) using numerical method. We use iteration method to solve (2.4) and (2.5) by representing these equations in the following form.

\[
\hat{c} = d \left[ \sum_{i=1}^{d} \frac{k(R_i+1)+1}{(1+x_i^c)} x_i^c \log(x_i) + \frac{kR_d^* t^c \log(t)}{(1+t^c)} - \sum_{i=1}^{d} \log(x_i) \right]^{-1}
\]

(2.6)

and

\[
\hat{k} = d \left[ \sum_{i=1}^{d} (R_i+1) \log(1+x_i^c) + R_d^* \log(1+t^c) \right]^{-1}.
\]

(2.7)

After obtaining the value of \( \hat{c} \) and \( \hat{k} \), we can obtain ML estimate of reliability \( \hat{F}(t_o) \) by using the invariance property of ML estimates as follows.

\[
\hat{F}(t_o) = (1 + t_o^c)^{-\hat{k}}.
\]

(2.8)

**Remarks:** When \( x_m \leq t \), we have \( d = m \) and \( R_d^* = 0 \). Which implies that all the expressions for Case II can be obtained by substituting \( d = m \) into the expression for case I.
Table 1  Removal patterns of units during lifetest for \( n=50 \) and different values of \( m \).

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( S_{1:1} )</th>
<th>( S_{1:2} )</th>
<th>( S_{1:3} )</th>
<th>( S_{1:4} )</th>
<th>( S_{1:5} )</th>
<th>( S_{1:6} )</th>
<th>( S_{2:1} )</th>
<th>( S_{2:2} )</th>
<th>( S_{2:3} )</th>
<th>( S_{2:4} )</th>
<th>( S_{2:5} )</th>
<th>( S_{2:6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m=20 )</td>
<td>( 0^*19,30^*1 )</td>
<td>( 30^*1,0^*19 )</td>
<td>( 0^*5,3^*10,0^*5 )</td>
<td>( (0.3)^*10 )</td>
<td>( 3^*10,0^*10 )</td>
<td>( 0^*10,3^*10 )</td>
<td>( 0^*39,10^*1 )</td>
<td>( 10^*1,0^*39 )</td>
<td>( 0^*15,1^*10,0^*15 )</td>
<td>( (0,0,0,1)^*10 )</td>
<td>( 1^*10,0^*30 )</td>
<td>( 0^*30,1^*10 )</td>
</tr>
<tr>
<td>( m=40 )</td>
<td>( 0^*19,30^*1 )</td>
<td>( 30^*1,0^*19 )</td>
<td>( 0^*5,3^*10,0^*5 )</td>
<td>( (0.3)^*10 )</td>
<td>( 3^*10,0^*10 )</td>
<td>( 0^*10,3^*10 )</td>
<td>( 0^*39,10^*1 )</td>
<td>( 10^*1,0^*39 )</td>
<td>( 0^*15,1^*10,0^*15 )</td>
<td>( (0,0,0,1)^*10 )</td>
<td>( 1^*10,0^*30 )</td>
<td>( 0^*30,1^*10 )</td>
</tr>
</tbody>
</table>

Here \( a^*b \) denotes \((a,a,a,\ldots,b\ times)\).

Table 2  Average estimates of parameters and their MSEs (in parenthesis) for various schemes at \( t=1.0 \).

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( \hat{c} )</th>
<th>( k )</th>
<th>( \hat{c}_s )</th>
<th>( k_s )</th>
<th>( \hat{c}_L )</th>
<th>( k_L )</th>
<th>( \hat{c}_L )</th>
<th>( k_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{1:1} )</td>
<td>3.741 0.879 3.556 0.836 3.335 0.816 3.687 0.844</td>
<td>(0.375) (0.046) (0.275) (0.013) (0.384) (0.014) (0.288) (0.014)</td>
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</tr>
<tr>
<td>( S_{1:2} )</td>
<td>3.62 0.852 3.431 0.797 3.144 0.779 3.616 0.804</td>
<td>(0.479) (0.067) (0.353) (0.014) (0.595) (0.019) (0.349) (0.015)</td>
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</tr>
<tr>
<td>( S_{1:3} )</td>
<td>3.701 0.878 3.509 0.818 3.285 0.798 3.644 0.825</td>
<td>(0.404) (0.057) (0.313) (0.014) (0.450) (0.015) (0.314) (0.014)</td>
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</tr>
<tr>
<td>( S_{1:4} )</td>
<td>3.695 0.878 3.509 0.825 3.285 0.803 3.643 0.833</td>
<td>(0.385) (0.057) (0.299) (0.014) (0.433) (0.016) (0.301) (0.014)</td>
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<tr>
<td>( S_{1:5} )</td>
<td>3.664 0.872 3.473 0.804 3.239 0.784 3.616 0.812</td>
<td>(0.425) (0.061) (0.335) (0.013) (0.498) (0.018) (0.330) (0.014)</td>
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<tr>
<td>( S_{1:6} )</td>
<td>3.711 0.872 3.523 0.829 3.302 0.808 3.655 0.837</td>
<td>(0.382) (0.047) (0.292) (0.014) (0.417) (0.016) (0.296) (0.014)</td>
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</tr>
<tr>
<td>( S_{2:1} )</td>
<td>3.724 0.865 3.555 0.847 3.350 0.826 3.671 0.855</td>
<td>(0.337) (0.034) (0.258) (0.010) (0.232) (0.011) (0.268) (0.009)</td>
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</tr>
<tr>
<td>( S_{2:2} )</td>
<td>3.697 0.865 3.519 0.841 3.296 0.818 3.650 0.849</td>
<td>(0.365) (0.042) (0.281) (0.011) (0.280) (0.014) (0.286) (0.011)</td>
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<tr>
<td>( S_{2:3} )</td>
<td>3.726 0.862 3.552 0.843 3.345 0.822 3.671 0.850</td>
<td>(0.346) (0.036) (0.265) (0.010) (0.252) (0.013) (0.275) (0.010)</td>
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</tr>
<tr>
<td>( S_{2:4} )</td>
<td>3.720 0.868 3.548 0.846 3.342 0.825 3.666 0.854</td>
<td>(0.344) (0.037) (0.265) (0.011) (0.264) (0.013) (0.274) (0.011)</td>
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</tr>
<tr>
<td>( S_{2:5} )</td>
<td>3.706 0.865 3.530 0.840 3.318 0.818 3.652 0.848</td>
<td>(0.360) (0.041) (0.280) (0.010) (0.275) (0.013) (0.286) (0.010)</td>
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</tr>
<tr>
<td>( S_{2:6} )</td>
<td>3.716 0.861 3.547 0.845 3.342 0.824 3.663 0.853</td>
<td>(0.348) (0.034) (0.269) (0.011) (0.255) (0.013) (0.278) (0.011)</td>
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</tr>
</tbody>
</table>
Table 3 Average estimates of parameters and their MSEs (in parenthesis) for various schemes at $t = 1.5$.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$\hat{c}$</th>
<th>$\hat{k}$</th>
<th>$\tilde{c}$</th>
<th>$\tilde{k}$</th>
<th>$\hat{c}_L$</th>
<th>$\hat{k}_L$</th>
<th>$\hat{c}_L$</th>
<th>$\hat{k}_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{1:1}$</td>
<td>3.769</td>
<td>0.886</td>
<td>3.606</td>
<td>0.841</td>
<td>3.409</td>
<td>0.82</td>
<td>3.715</td>
<td>0.848</td>
</tr>
<tr>
<td></td>
<td>(0.330)</td>
<td>(0.041)</td>
<td>(0.243)</td>
<td>(0.013)</td>
<td>(0.305)</td>
<td>(0.013)</td>
<td>(0.263)</td>
<td>(0.013)</td>
</tr>
<tr>
<td>$S_{1:2}$</td>
<td>3.708</td>
<td>0.878</td>
<td>3.553</td>
<td>0.829</td>
<td>3.329</td>
<td>0.806</td>
<td>3.678</td>
<td>0.838</td>
</tr>
<tr>
<td></td>
<td>(0.364)</td>
<td>(0.047)</td>
<td>(0.274)</td>
<td>(0.014)</td>
<td>(0.376)</td>
<td>(0.014)</td>
<td>(0.288)</td>
<td>(0.014)</td>
</tr>
<tr>
<td>$S_{1:3}$</td>
<td>3.753</td>
<td>0.883</td>
<td>3.586</td>
<td>0.836</td>
<td>3.384</td>
<td>0.814</td>
<td>3.694</td>
<td>0.844</td>
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<tr>
<td></td>
<td>(0.338)</td>
<td>(0.043)</td>
<td>(0.256)</td>
<td>(0.013)</td>
<td>(0.332)</td>
<td>(0.013)</td>
<td>(0.269)</td>
<td>(0.014)</td>
</tr>
<tr>
<td>$S_{1:4}$</td>
<td>3.778</td>
<td>0.889</td>
<td>3.597</td>
<td>0.843</td>
<td>3.387</td>
<td>0.823</td>
<td>3.718</td>
<td>0.851</td>
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<tr>
<td></td>
<td>(0.358)</td>
<td>(0.043)</td>
<td>(0.255)</td>
<td>(0.013)</td>
<td>(0.331)</td>
<td>(0.013)</td>
<td>(0.278)</td>
<td>(0.014)</td>
</tr>
<tr>
<td>$S_{1:5}$</td>
<td>3.743</td>
<td>0.88</td>
<td>3.58</td>
<td>0.831</td>
<td>3.38</td>
<td>0.809</td>
<td>3.691</td>
<td>0.839</td>
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<tr>
<td></td>
<td>(0.339)</td>
<td>(0.046)</td>
<td>(0.259)</td>
<td>(0.013)</td>
<td>(0.335)</td>
<td>(0.013)</td>
<td>(0.273)</td>
<td>(0.013)</td>
</tr>
<tr>
<td>$S_{1:6}$</td>
<td>3.785</td>
<td>0.888</td>
<td>3.609</td>
<td>0.842</td>
<td>3.406</td>
<td>0.822</td>
<td>3.723</td>
<td>0.85</td>
</tr>
<tr>
<td></td>
<td>(0.341)</td>
<td>(0.041)</td>
<td>(0.245)</td>
<td>(0.013)</td>
<td>(0.311)</td>
<td>(0.013)</td>
<td>(0.267)</td>
<td>(0.014)</td>
</tr>
</tbody>
</table>

3 Bayesian Estimation

According to Bayesian paradigm, we consider parameters $c$ and $k$ to be random variables. Let $\pi(c)$ and $\pi(k)$, respectively, denoting the prior densities for $c$ and $k$, follow gamma distributions with respective densities given by,

$$\pi(c) \propto c^{b-1}e^{-ac} ; \quad a > 0, b > 0, c > 0 \quad (3.1)$$

and

$$\pi(k) \propto k^{p-1}e^{-qk} ; \quad p > 0, q > 0, k > 0. \quad (3.2)$$

Assuming $c$ and $k$ statistically independent, the joint prior distribution of $c$ and $k$ can be written as

$$\pi(c, k) \propto c^{b-1}k^{p-1} \exp\{-acqk\} ; \quad a, b, q, p, c, k > 0 \quad (3.3)$$

Merging the likelihood function (2.3) with joint prior distribution of $c$ and $k$, we obtain the joint posterior density of $c$ and $k$, given the data, as follows.

$$\Pi(c, k|x) \propto c^{d+b-1}k^{d+p-1}\prod_{i=1}^{d}\left\{x_i^{-c} (1+x_i)^{-(kR_i+1)}\right\}(1+c) e^{-kr_i}\exp\{-acqk\} \quad (3.4)$$
Table 4  Average estimates of reliability function at \((t_0 = 1.0)\) and their MSEs×10^{-2} (in paranthesis) for various schemes.

<table>
<thead>
<tr>
<th>Time</th>
<th>Scheme</th>
<th>(R(t))</th>
<th>(R_S(t))</th>
<th>(R_L(t))</th>
<th>(R_L(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(t = 1)</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>(t = 1.5)</td>
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<tr>
<td></td>
<td>((\nu_1 = -0.5))</td>
<td>((\nu_2 = 1.5))</td>
<td>((\nu_1 = -0.5))</td>
<td>((\nu_2 = 1.5))</td>
<td></td>
</tr>
<tr>
<td>(S_{1:1})</td>
<td>0.549</td>
<td>0.559</td>
<td>0.556</td>
<td>0.571</td>
<td>0.549</td>
</tr>
<tr>
<td></td>
<td>(0.626)</td>
<td>(0.256)</td>
<td>(0.196)</td>
<td>(0.198)</td>
<td>(0.390)</td>
</tr>
<tr>
<td>(S_{1:2})</td>
<td>0.563</td>
<td>0.563</td>
<td>0.559</td>
<td>0.575</td>
<td>0.551</td>
</tr>
<tr>
<td></td>
<td>(0.992)</td>
<td>(0.281)</td>
<td>(0.208)</td>
<td>(0.207)</td>
<td>(0.471)</td>
</tr>
<tr>
<td>(S_{1:3})</td>
<td>0.551</td>
<td>0.562</td>
<td>0.558</td>
<td>0.574</td>
<td>0.551</td>
</tr>
<tr>
<td></td>
<td>(0.776)</td>
<td>(0.266)</td>
<td>(0.194)</td>
<td>(0.198)</td>
<td>(0.417)</td>
</tr>
<tr>
<td>(S_{1:4})</td>
<td>0.554</td>
<td>0.561</td>
<td>0.558</td>
<td>0.573</td>
<td>0.551</td>
</tr>
<tr>
<td></td>
<td>(0.696)</td>
<td>(0.266)</td>
<td>(0.199)</td>
<td>(0.203)</td>
<td>(0.409)</td>
</tr>
<tr>
<td>(S_{1:5})</td>
<td>0.554</td>
<td>0.562</td>
<td>0.558</td>
<td>0.574</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>(0.846)</td>
<td>(0.274)</td>
<td>(0.197)</td>
<td>(0.202)</td>
<td>(0.444)</td>
</tr>
<tr>
<td>(S_{1:6})</td>
<td>0.553</td>
<td>0.56</td>
<td>0.556</td>
<td>0.572</td>
<td>0.549</td>
</tr>
<tr>
<td></td>
<td>(0.656)</td>
<td>(0.261)</td>
<td>(0.196)</td>
<td>(0.198)</td>
<td>(0.404)</td>
</tr>
<tr>
<td>(S_{2:1})</td>
<td>0.554</td>
<td>0.564</td>
<td>0.56</td>
<td>0.577</td>
<td>0.545</td>
</tr>
<tr>
<td></td>
<td>(0.477)</td>
<td>(0.232)</td>
<td>(0.191)</td>
<td>(0.198)</td>
<td>(0.318)</td>
</tr>
<tr>
<td>(S_{2:2})</td>
<td>0.554</td>
<td>0.57</td>
<td>0.565</td>
<td>0.582</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>(0.644)</td>
<td>(0.287)</td>
<td>(0.166)</td>
<td>(0.176)</td>
<td>(0.591)</td>
</tr>
<tr>
<td>(S_{2:3})</td>
<td>0.548</td>
<td>0.567</td>
<td>0.562</td>
<td>0.58</td>
<td>0.555</td>
</tr>
<tr>
<td></td>
<td>(0.568)</td>
<td>(0.280)</td>
<td>(0.155)</td>
<td>(0.17)</td>
<td>(0.513)</td>
</tr>
<tr>
<td>(S_{2:4})</td>
<td>0.546</td>
<td>0.565</td>
<td>0.561</td>
<td>0.577</td>
<td>0.553</td>
</tr>
<tr>
<td></td>
<td>(0.542)</td>
<td>(0.275)</td>
<td>(0.163)</td>
<td>(0.175)</td>
<td>(0.514)</td>
</tr>
<tr>
<td>(S_{2:5})</td>
<td>0.552</td>
<td>0.569</td>
<td>0.564</td>
<td>0.581</td>
<td>0.554</td>
</tr>
<tr>
<td></td>
<td>(0.639)</td>
<td>(0.298)</td>
<td>(0.154)</td>
<td>(0.173)</td>
<td>(0.571)</td>
</tr>
<tr>
<td>(S_{2:6})</td>
<td>0.551</td>
<td>0.564</td>
<td>0.56</td>
<td>0.577</td>
<td>0.555</td>
</tr>
<tr>
<td></td>
<td>(0.566)</td>
<td>(0.284)</td>
<td>(0.158)</td>
<td>(0.177)</td>
<td>(0.484)</td>
</tr>
</tbody>
</table>

Table 5  Sample observations, drawn from the considered real data, under various schemes.

<table>
<thead>
<tr>
<th>Schemes</th>
<th>Sample Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1 = 0 \times 11)</td>
<td>0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01</td>
</tr>
<tr>
<td>(S_2 = 7.0 \times 11)</td>
<td>0.19, 1.31, 3.16, 4.15, 4.85, 6.50, 7.35, 12.06, 31.75, 32.52</td>
</tr>
<tr>
<td>(S_3 = 0 \times 4.2 \times 3, 1.0 \times 4)</td>
<td>0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 6.50, 8.01, 8.27, 31.75, 32.52</td>
</tr>
<tr>
<td>(S_4 = (0.1) \times 5.0, 2)</td>
<td>0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 7.35, 8.01, 8.27, 32.52</td>
</tr>
<tr>
<td>(S_5 = 1 \times 7.0 \times 5)</td>
<td>0.19, 0.96, 1.31, 2.78, 3.16, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06</td>
</tr>
<tr>
<td>(S_6 = 0 \times 5.1 \times 7)</td>
<td>0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.85, 7.35, 8.01, 8.27, 32.52</td>
</tr>
</tbody>
</table>

Under the SELF, the Bayes estimate of any parametric function \(\omega (\theta)\) (say) can be obtained using the following expression.

\[
E (\omega (\theta) \mid x) = \frac{\int_0^\infty \omega (\theta) \Pi (\theta \mid x) d\theta}{\int_0^\infty \Pi (\theta \mid x) d\theta}.
\] (3.5)

To obtain Bayes estimators of \(c\) and \(k\) using (3.5), we need the marginal posterior densities of these parameter. However from (3.4), we observe that it is not possible to obtain the marginal posterior in explicit form. Therefore we evaluate these estimates using Lindely’s approximation.
According to Lindley’s approximation [16], the posterior expectation of any parametric function \( \omega(\theta) = \omega(c, k) \), which is a ratio of two integrals given by (3.5), can be obtained in the form of the following expression.

\[
\hat{\omega}(c, k) = \hat{\omega}(c, k) + \frac{1}{2} [A + l_{30} B_{12} + l_{05} B_{21} + l_{21} C_{12} + l_{12} C_{21}] + \rho_1 A_{12} + \rho_2 A_{21},
\]

(3.6)

where

\[
A = \sum_{i=1}^{2} \sum_{j=1}^{2} \omega_{ij} \sigma_{ij}, \quad l = \log(L), \quad l_{\eta \zeta} = \frac{d^{\eta + \zeta}}{dc^{\eta}dk^{\zeta}}, \quad \eta + \zeta = 0, 1, 2, 3; \text{ with } \eta + \zeta = 3 \text{ for } i, j = 1, 2.
\]

\[
\rho_1 = \frac{\partial \omega}{\partial c}, \quad \rho_2 = \frac{\partial \omega}{\partial k}, \quad \omega_1 = \frac{\partial \omega}{\partial c}, \quad \omega_2 = \frac{\partial \omega}{\partial k}, \quad \omega_{12} = \frac{\partial^2 \omega}{\partial c \partial k}, \quad \omega_{21} = \frac{\partial^2 \omega}{\partial k \partial c}, \quad \omega_{11} = \frac{\partial^2 \omega}{\partial c^2}, \quad \omega_{22} = \frac{\partial^2 \omega}{\partial k^2},
\]

where \( \rho = \log \pi(c, k) \) and for \( i \neq j \), \( A_{ij} = \omega_i \sigma_{ji} + \omega_j \sigma_{ij}, \quad B_{ij} = (\omega_i \sigma_{ji} + \omega_j \sigma_{ij}) \sigma_{ii} \) and \( C_{ij} = 3 \omega_i \sigma_{ji} \sigma_{ij} + \omega_j (\sigma_{ij} \sigma_{jj} + 2 \sigma_{ij}^2) \).

Here \( \sigma_{ij} \) is the \( (i, j)^{th} \) element in the inverse of the matrix \( \{-l_{ij}\}; i, j = 1, 2, \ldots \).

Let \( \sigma_{11} = \frac{H}{N}, \quad \sigma_{22} = \frac{G}{N}, \quad \sigma_{12} = \sigma_{21} = -\frac{I}{N}, \) where \( N = GH - I^2 \).

With these notations and using the expression given by [17], (3.6) can be written as follows,

\[
E(\omega(\theta)|\chi) = \hat{\omega}(\theta) + \omega_1 \psi_1 + \omega_2 \psi_2 + \phi,
\]

(3.7)

where

\[
\psi_1 = \frac{1}{N} (H \rho_1 - I \rho_2) + \frac{1}{2N^2} \left[H^2 l_{30} - I G l_{03} + (GH + 2I^2) l_{12} - 3I H l_{21}\right],
\]

\[
\psi_2 = \frac{1}{N} (G \rho_2 - I \rho_1) + \frac{1}{2N^2} \left[G^2 l_{03} - I H l_{30} + (GH + 2I^2) l_{21} - 3I G l_{12}\right]
\]

and \( \phi = \frac{1}{2N} [H \omega_{11} - I (\omega_{12} + \omega_{21}) + G \omega_{22}] \).

All expressions in (3.7) are to be evaluated at \( c = \hat{c} \) and \( k = \hat{k} \).

For our estimation problem, we have
where, \( \psi \) and \( \omega \) when

3.2 Bayes Estimator of \( k \)

In order to obtain the Bayes estimator of any parametric function of \( c \) and \( k \). We substitute it for \( \omega(\theta) \) in (3.5). With their notation, we get the Bayes estimates of parameters and reliability function under SELF as follows.

3.1 Bayes Estimator of \( c \)

When \( \omega(\theta) = c \), we have \( \omega_1 = 1, \omega_2 = 0 \) and \( \phi = 0 \). Substituting these values in (3.7), we get the Bayes estimator of \( c \) under SELF as follows

\[
\tilde{c}_s = \hat{c} + \psi_1. \tag{3.8}
\]

3.2 Bayes Estimator of \( k \)

When \( \omega(\theta) = k \), we get \( \omega_1 = 0, \omega_2 = 1 \) and \( \phi = 0 \). Then from (3.7), we get the Bayes estimator of \( k \) under SELF as follows

\[
\tilde{k}_s = \hat{k} + \psi_2. \tag{3.9}
\]

3.3 Bayes Estimator of Reliability function

When \( \omega(\theta) = R(t) = (1 + t^c)^{-k} \), using (3.7) the Bayes estimator of \( R(t) \) comes out to be

\[
\hat{R}(t) = R(t) + \omega_1 \psi_1 + \omega_2 \psi_2 + \phi, \tag{3.10}
\]

where,

\[
\omega_1 = -k(1 + t^c)^{-(k+1)}t^c \log(t)
\]

and

\[
\omega_2 = -(1 + t^c)^{-k} \log(1 + t^c).
\]

Further \( \psi_1, \psi_2 \) and \( \phi \) can be evaluated from (3.7) by using

\[
\omega_{11} = -kt^c(1 + t^c)^{-(k+2)}(1 - k t^c) \log^2(t),
\]

\[
\omega_{22} = (1 + t^c)^{-k} \log^2(1 + t^c)
\]

and

\[
\omega_{12} = \omega_{21} = -t^c(1 + t^c)^{-(k+1)}(1 - k \log(1 + t^c)) \log(t).
\]
Reliability Estimation for Burr type-XII Distribution Under Type I Progressive Hybrid Censoring Scheme

4 Bayes estimation under LINEX loss function

The LINEX loss function, introduced by [18], is defined as follows

\[ L(\hat{\theta}, \theta) = Se^{V\Delta} - S\Delta - t, \] (4.1)

where S and V are shape and scale parameters of the loss function, respectively. Here we take the value of S to be 1 and \( \Delta = (\hat{\theta} - \theta) \) denotes the scalar estimation error in using \( \hat{\theta} \) to estimate \( \theta \). The Bayes estimate of any parametric function \( \omega(\theta) \) under LINEX loss function [for details, see [19]] is given by

\[ \tilde{\omega}_L(\theta) = -\frac{1}{V} \log \left[ E \left( \exp(-V\omega(\theta)) \right) \right], \] (4.2)

provided \( E \left( \exp(-V\omega(\theta)) \right) \) exists and finite. To obtain \( \tilde{\omega}_L(\theta) \) the Bayes estimate of \( \omega(\theta) \) under LINEX loss function, we first evaluate the posterior expectation \( E \left( \exp(-V\omega(\theta)) \right) \) for given \( V \) using (3.7), then the Bayes estimates of \( c, k \) and reliability function can be obtained as follows.

4.1 Bayes Estimator of \( c \)

When \( \omega(\theta) = \exp(-Vc) \), using (3.7) and (4.2), we get the following expressions to obtain the Bayes estimate of \( c \) under LINEX loss function.

\[ \tilde{c}_L = -\frac{1}{V} \log \left[ \exp(-Vc) \right] \]

\[ = -\frac{1}{V} \log \left[ \exp(-Vc) + \omega_1\psi_1 + \phi_c \right], \] (4.3)

where, \( \omega_1 = -V \exp(-Vc) \) and \( \phi_c = \left( HV^2 \exp(-Vc) \right) / (2N) \).

4.2 Bayes Estimator of \( k \)

When \( \omega(\theta) = \exp(-Vk) \), the Bayes estimate of \( k \) is given by

\[ \tilde{k}_L = -\frac{1}{V} \log \left[ \exp(-Vk) \right] \]

\[ = -\frac{1}{V} \log \left[ \exp(-Vk) + \omega_2\psi_2 + \phi_k \right], \] (4.4)

where, \( \omega_2 = -V \exp(-Vk) \) and \( \phi_k = \left( GV^2 \exp(-Vk) \right) / (2N) \).

4.3 Bayes Estimator of Reliability function

When we put \( \omega(\theta) = R(t) = (1+t^c)^{-k} \) in (3.7), then the Bayes estimate of \( R(t) \) is given as follows.

\[ \tilde{R}_L(t) = -\frac{1}{V} \log \left[ \exp \left( -V(1+t^c)^{-k} \right) \right] \]

\[ = -\frac{1}{V} \log \left[ \exp \left( -V(1+t^c)^{-k} \right) + \omega_1\psi_1 + \omega_2\psi_2 + \phi_{R(t)} \right], \] (4.5)

where, \( \omega_1 = Vkt^c \log(t)(1+t^c)^{-(k+1)} \exp(-V(1+t^c)^{-k}) \) and
Further, $\psi_1$, $\psi_2$ and $\phi$ can be evaluated from (3.7) by using
\[
\omega_1 = kVt^c \log^2(t)(1+t^c)^{(k+1)} \exp(-V(1+t^c)^{-k}) \{1 + kVt^c(1+t^c)^{-(k+1)} - t^c(k+1)(1+t^c)^{-1}\},
\]
\[
\omega_2 = V(1+t^c)^{-k} \log^2(1+t^c) \exp(-V(1+t^c)^{-k}) \{V(1+t^c)^{-k} - 1\}
\]
and
\[
\omega_{12} = \omega_{21} = t^c V(1+t^c)^{-(k+1)} \log(t) \exp(-V(1+t^c)^{-k}) \{1 - k \log(1+t^c) + kV(1+t^c)^{-k} \log(1+t^c)\}.
\]

5 Simulation Study

In this section we draw some inferences based on simulated data. For this study we generate samples from Burr XII model under type I PHCS, for the value of parameters $c = 3.7$ and $k = 0.85$, by using algorithm given by [20]. The R software is used for the entire computation. For this study we decide $n = 50$ and generate samples for $m = 20$ and $m = 40$ by choosing various removals patterns of observations. The schemes based on these removal patterns are presented in Table 1. We generate 3000 samples for each case and evaluate the values of various estimates based on these.

Using the generated data, we obtain ML estimates for parameters $c$ and $k$ and $R(t)$. For Bayesian study, we choose values of hyperparameters to be $a = 1.2$, $b = 3$, $p = 8$ and $q = 10$. With these values we obtain Bayes estimates of parameters using Lindley’s approximation under SELF and LINEX loss functions. The estimated values of $c$ and $k$ and their mean squared errors (MSEs) are given in Table 2 and that of the reliability function along with its MSEs are given in Table 3. We take $V = 1.5$ while presenting the case of overestimation and $V = -0.5$ for underestimation. It can be seen from Tables 2, 3 and 4 that MSEs of all the estimates decreases as $m$ increases. The decreasing behaviour of MSE is also seen as $t$, the time of termination of experiment, increases.

6 Real Data Analysis and Conclusion

Here, we analyze a data set reported by [21]. The data represents the times (in minutes) to breakdown of an insulating fluid between electrodes at the voltage of 34 KV. The 19 times to breakdown are

0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50,
7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.

We have first used Kolmogorov-Smirnov (K-S) test to check whether the data fit the distribution or not. By executing K-S test with R software, the p-value comes out to be 0.262 which shows that the distribution is quite a good fit for the data. Using this data, we fix $m = 12$, $t = 33$ minutes and obtain different samples according to various choice of the removal patterns of observations which are presented in Table 5. We first obtain the MLEs of parameters and reliability function for all the samples and present the same in Table 6. Since we do not have any prior information about the values of these parameters, we use non-informative prior for Bayesian estimation. It is to mention here that non-informative priors for $c$ and $k$ can be obtained from (3.1) and (3.2), respectively, by choosing the values of the hyper-parameters to be $a = b = p = q = 0$. For these values, the values of Bayes estimates of parameter and reliability function are given in Table 6.
Conclusion

In this paper, we provided procedures to obtain ML estimates of parameters and reliability function of Burr type XII distribution using type I progressively hybrid censored data. We first obtained the MLEs of parameters then using these MLEs, we evaluated the Bayes estimates of the parameters and reliability function by applying Lindleys approximation. Through simulation study, we have shown the performance of various schemes in terms of MSEs of estimates of parameters. In order to present the practical applications of the study, we have analyzed a real data set. Beside using the recent concept of progressive hybrid censoring, we performed Bayesian computations through Lindleys approximation. Therefore the study may benefit the researchers who are not much familiar with advanced computational techniques.

In this paper, we have worked with the type I PHC scheme based on random removals of observations during life test. The work may further be extended by considering any appropriate distribution for the removal patterns.

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References


