Nonlinear dynamics of mini-satellite respinup by weak internal control torques

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Abstract. Contemporary space engineering advanced new problem before theoretical mechanics and motion control theory: a spacecraft directed respinup by the weak restricted control internal forces. The paper presents some results on this problem, which is very actual for energy supply of information mini-satellites (for communication, geodesy, radio- and opto-electronic observation of the Earth et al.) with electro-reaction plasma thrusters and gyro moment cluster based on the reaction wheels or the control moment gyros. The solution achieved is based on the methods for synthesis of nonlinear robust control and on rigorous analytical proof for the required spacecraft rotation stability by Lyapunov function method. These results were verified by a computer simulation of strongly nonlinear oscillatory processes at respinuping of a flexible spacecraft.

1 Introduction

Recently, space science and engineering advanced new problem before theoretical mechanics, physics and motion control theory: a spacecraft directed respinup by the weak restricted control internal forces, which close to famous question of Archimedes: is it possible to turn over the Earth without any external support point? Academician Vladimir Matrosov [1] (08.05.1932 – 17.04.2011) called the problem as "Titov task" because of his pupil G. Titov, principal researcher of "Reshetnev ISS" space company. The paper presents some results on this problem, which has large significance for engineering practice, for example it is very actual for energy supplying of the communication mini-satellites with plasma thrusters at initial mission modes.

In current practice the information mini-satellites are equipped with thruster unit based on plasma reaction thrusters (RTs) having high specific pulse and large power consumption. While designing a
mini-satellite weighted of 100 to 500 kg it is very attractive to employ plasma RTs only for all modes. At the problem the constrains are as follows [2]:

- On separating from a launcher, a spacecraft (SC) obtains an initial angular rate up to $20^\circ/s$. During that SC rotation an electric power required for the onboard equipment is generated by solar arrays panels (SAPs) or by chemical batteries. An energy generated by the SAPs depends on an angle between their normal and direction towards the Sun.

- Plasma RT enjoy small thrust values (about several grams) and large power consumption (magnitude of 1 to 1.5 kW). Small thrusts and therefore small control torques are the cause of a long time period required to damp initial SC rate. The plasma RTs can be activated a specified time period $T_a$ from several hours to several days after the separation.

- Severe requirements applied to the mass of the attitude & orbit control system (AOCS) installed on a satellite result in the fact that the angular momentum (AM) of a gyro moment cluster (GMC) based on the reaction wheels (RWs) or on the single-gimbal control moment gyroscopes – gyrodines (GDs), is significantly lower then the spacecraft AM obtained after its separation.

The engineering problem is to ensure such motion of a SC separated with no plasma RTs used, under which the energetic conditions are met, and then after the specified period $T_a$ to complete a SC orientation towards the Sun by plasma RTs. The approach applied is based on two main assumptions:

1. plasma RTs are applied to perform two tasks: (i) satellite attitude control and unloading of an accumulated AM, and (ii) satellite orbit control;
2. small-mass GMC having a small AM is applied at initial mode without joining-up the RTs.

At a separation time moment $t_0$, the SC body AM vector $K_0 \equiv J\omega(t_0) = G_0$ has an arbitrary direction, therefore principal problem is to coincide this vector with the SC body’s maximum inertia axis Oy using only the GMC having small resources for the AM and control torque variation domains.

Essentially nonlinear dynamical processes are arising from a moving the total AM vector $G(t)$ of mechanical system with respect to the satellite body reference frame (BRF) Oxyz. Moreover, a Sun sensor is switched on, the Sun position is determined within the BRF and, if required, the SAPs are turned by an angle $\gamma^p$, $0 \leq \gamma^p \leq 270^\circ$. In result, the SC angular rate is set along the axis Oy which is perpendicular to the SAPs rotation axis. Depending on the initial vector $G$ angular position and...
direction \( S \) towards the Sun, the SAPs will be illuminated either continuously when the vectors \( G \) and \( S \) have coincided, or periodically if \( G \perp S \), see Fig. 1. At this phase of the SC mission, the GMC is applied to generate control torques and plasma RTs are not activated. At next phase of the AOCS initial modes the RTs are turned on and generate the control torques to damp a spacecraft angular rate.

2 The problem background

Most satellites contain a GMC to provide gyroscopic stability of a desired attitude of the SC body, problems of gyrostat optimal control [3] – [6] and synthesis of control laws had been studied by V.I. Zubov [7] – [9]. V.I. Zubov’s results were essentially developed by Ye.Ya. Smirnov [10] and his successors [11, 12]. Here a Lyapunov function is applied with small parameter for its crossed term. This idea for mechanical systems rises to G.I. Chetayev [13]. Instead that A.V. Yurkov [14] applied a large parameter for a position term at the Lyapunov function. The SC spinup problems have been investigated by numerous authors [16] – [20] et al. C.D. Hall [19] have been obtained a bifurcation diagram for all gyrostat spinup equilibria manifolds. Different approaches were applied to convert the intermediate axis spin equilibrium to those of major axis spin (to respinup spacecraft’s body) by variation of the RWs AM [15] – [21].

If enough AM is added, desired spin is globally stable in the presence of energy dissipation [17]. However, no literature was found suggesting the SC respinup feedback control by the GMC having small resources, when spacecraft’s body AM vector have a large value and an arbitrary direction.

In the paper, only principle aspects of strongly nonlinear dynamics related to the robust controlled coincidence of the SC body Oy axis with the SC’s AM vector \( G \) are presented. Results early obtained, see Fig. 3 in [22], are direct proofs for large efficiency of the GDs as compared with the RWs. The solution achieved is based on the methods for synthesis of nonlinear robust control [23, 24] and on rigorous analytical proof for the required SC rotation stability [25, 22]. These results were verified by computer simulation of strongly nonlinear oscillatory processes at respinuping a flexible spacecraft.

3 Mathematical models

Let us we have a free rigid body (RB) with one fixed point \( O \) and any GMC. An inertia tensor \( J \) of the RB with a GMC is a arbitrary diagonal one, i.e. \( J = [J_x, J_y, J_z] \equiv \text{diag} \{J_i\}, i=x,y,z \equiv 1 \div 3 \), within the BRF \( Oxyz \). Model of the RB motion is presented in well-known vector form

\[
\dot{\mathbf{K}} + \mathbf{\omega} \times \mathbf{G} = \mathbf{M} \equiv -\dot{\mathbf{H}},
\]

where \( \mathbf{\omega} = \{\omega_i\} \) is angular rate vector of the RB; \( \mathbf{K} = J \mathbf{\omega} \) is the AM vector of the RB equipped with a GMC; \( \mathbf{G} = \mathbf{K} + \mathbf{H} \) is a total AM for mechanical system in the whole; \( \mathbf{H} \) is a column vector of the GMC total AM. It is suitable to present any GMC type using a canonical reference frame (CRF) \( \mathbf{E}^E = (x_E^E, y_E^E, z_E^E) \). Necessary location of required domain \( \mathcal{S} \) of the GMC AM vector \( \mathbf{H} \) variation is achieved by the CRF orientation versus the BRF.

For any GMC based on 4 RWs having identical own axial inertia moment \( J_r \), model of the system motion can be presented by two vector equations

\[
\dot{\mathbf{K}} + \dot{\mathbf{H}} + \mathbf{\omega} \times \mathbf{G} = \mathbf{0}; \quad J_r \mathbf{A}_r^T \dot{\mathbf{\omega}} + \mathbf{H} = \mathbf{M}_r,
\]

where for \( p = 1 \div 4 \) vector column of the GMC total AM \( \mathbf{H} = \sum \mathbf{h}_p \equiv \sum \mathbf{e}_p \mathbf{h}_p = \mathbf{A}_r \mathbf{H} \); \( \mathbf{A}_r \) is a rectangular matrix; columns \( \mathbf{H} = \{\mathbf{h}_p\} \) and \( \mathbf{M}_r = \{m'_r\} \) are the own AMs and control torques applied along
the RW rotation axes. The AM vector \( \mathbf{G} = \mathbf{J} \omega + \mathbf{H} \) is represented as \( \mathbf{G} = \mathbf{K}_o + \dot{\mathbf{H}}_e \) with \( \mathbf{K}_o = \mathbf{J} \omega \), \( \mathbf{J}_e = \mathbf{J} - J_r \mathbf{A}_y \mathbf{A}_y^T \), \( \mathbf{H}_e = \mathbf{A}_y \mathbf{H}_r \) and \( \mathbf{H}_r = J_r \mathbf{A}_y^T \omega + \mathbf{H} \). Then for notation \( \mathbf{M}_e \equiv -\mathbf{A}_y \dot{\mathbf{H}}_e = -\mathbf{A}_y \mathbf{M}_r \) from (3.2) we obtain the SC motion model in the form

\[
\mathbf{K}_o + \omega \times \mathbf{G} = -\dot{\mathbf{H}}_e \equiv \mathbf{M}_e^o; \quad \dot{\mathbf{H}}_e = \mathbf{M}_r, \tag{3.3}
\]

where first equation has a structure of (3.1). At other hand, if we assume in (3.2) vector column \( \mathbf{M} = \mathbf{M}_r = -\mathbf{H} \equiv -\mathbf{A}_y \mathbf{H} \) to be known, then for complete compliance with (3.1) evident definition \( \mathbf{H} \) enables to calculate vector \( \mathbf{M}_r = \mathbf{H} + J_r \mathbf{A}_y^T \mathbf{h} \) of required RW control torque vector in the form \( \mathbf{M}_r = H + J_r \mathbf{A}_y^T \mathbf{J}^{-1} (\mathbf{M}^o - \mathbf{G} \times \omega) \). Obviously, with an arbitrary matrix \( \mathbf{A}_y \) these constrains are converted into fixed convex domains of allowable variation for the AM vector \( \mathbf{H} = \mathbf{A}_y \mathbf{H} \) and the control torque vector \( \mathbf{M} = -\mathbf{A}_y \mathbf{M}_r \) attributed to the GMC type.

For each RW the control torque and own AM are limited as per a module, i.e. \( \forall t \in T_0 \equiv [t_0, \infty) \)

\[
|m_p(t)| \leq m^m; \quad |h_p(t)| \leq h^m, \quad p = 1 \div 4, \tag{3.4}
\]

where parameters \( m^m \) and \( h^m \) are specified positive constants. We introduce normed AM \( h_p = h_p/h^m \) with \( |h_p| \leq 1 \) and notations \( S_\gamma \equiv \sin \gamma^w, C_\gamma \equiv \cos \gamma^w \) for the RW General Electric (GE) scheme which is presented in Fig. 2 at the normalized to \( h^m \) form. At notations

\[
x_1 = C_\gamma (h_1 + h_2); \quad x_2 = C_\gamma (h_3 + h_4); \quad x = x_1 + x_2; \quad y = S_\gamma (h_1 - h_2); \quad z = S_\gamma (h_3 - h_4);
\]

we have normed AM vector \( \mathbf{h} = \sum \mathbf{h}_p \equiv \sum \mathbf{e}_p h_p = \mathbf{H}/h^m \equiv \mathbf{A}_y \mathbf{H}/h^m \equiv \{x,y,z\} \) where matrix

\[
\mathbf{A}_y = \begin{bmatrix} C_\gamma & C_\gamma & C_\gamma & C_\gamma \\ S_\gamma & -S_\gamma & 0 & 0 \\ 0 & 0 & S_\gamma & -S_\gamma \end{bmatrix}.
\]

Applied 2-SPE (2 Scissored Pair Ensemble) scheme on 4 GDs with own AM \( h_g \) is presented in Fig. 3. Here within the CMG precession theory the AM vector \( \mathbf{H} = \mathbf{h}_g \mathbf{A}_y \mathbf{h} \) with constant non-singular matrix \( \mathbf{A}_y \), where a normed vector \( \mathbf{h} = \sum \mathbf{h}_p (\beta_p) \) made up from units \( \mathbf{h}_p (\beta_p) \), vector column \( \beta = \{\beta_p\} \) presents the GD’s angles, at last vector column \( \mathbf{h} \equiv \{x,y,z\} \), where \( x = x_{12} + x_{34}; \quad x_{12} = x_1 + x_2; \quad x_{34} = x_3 + x_4; \quad y = y_1 + y_2; \quad z = -(z_3 + z_4); \quad x_p = Cp_p; \quad y_p = Sp_p; \quad z_p = Sp_p.

At the command column \( \mathbf{u} = \{u_p\} \) vector of the GMC output torque has the form

\[
\mathbf{M}^o = -\dot{\mathbf{H}} (\beta, \dot{\beta}) = -h_g \mathbf{A}_h (\beta) \mathbf{u}; \quad \dot{\beta} = \mathbf{u}, \tag{3.5}
\]
where \( A_0(\beta) = A_\gamma A_h(\beta) \) and for notations \( S_\gamma \equiv \sin \gamma, \ C_\gamma \equiv \cos \gamma \) matrices \( A_\gamma \) and \( A_h(\beta) = \partial h(\beta) / \partial \beta \) are presented as follows

\[
A_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & S_\gamma & S_\gamma \\ 0 & -C_\gamma & C_\gamma \end{bmatrix} \quad A_h(\beta) = \begin{bmatrix} -y_1 & -y_2 & -z_3 & -z_4 \\ x_1 & x_2 & 0 & 0 \\ 0 & 0 & -x_3 & -x_4 \end{bmatrix}.
\]

The GDs’ angles vary within full range, but domain \( S_{SSS} \) of the GMC’s AM vector \( \mathbf{H}(\beta) \) variations is limited. The “control” \( u_p(t) \) of each GD is module-limited by given positive parameter \( u^m \):

\[
|u_p(t)| \leq u^m, \quad p = 1 \div 4, \quad \forall t \in T_0.
\]

These constrains are converted into \( \beta \)-dependent convex variation domain \( \mathcal{M} \) for a control torque \( M = M^g = -\mathbf{H}(\beta, \dot{\beta}) \) in the model (3.5).

With standard notations model of a free flexible SC motion is presented in vector-matrix form

\[
\begin{bmatrix} J & D_q \\ D_q^T & A_q \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} M - \omega \times G \\ -\{a_q (\bar{\Omega}_q^q (\Omega_j^q)^2 q_j)\} \end{bmatrix} ; \quad A^q = \{a_q^{\dot{q}}\} ; \quad G = G^o + D_q \dot{q}; \quad q = \{q_j\}; \quad G^o = J \omega + \mathbf{H}(\beta).
\]

\( 4 \) The problem statement

Considering model (3.1), let denote an AM vector of a RB at initial time moment \( t_0 \) as \( \mathbf{K}_0 \). Let vector of a GMC’s total AM at initial time be equal to zero, i.e. \( \mathbf{H}_0 \equiv \mathbf{H}(t_0) = 0 \). A norm of vector \( \mathbf{K}_0 \) is assumed to be limited with given constant, i.e. \( \| \mathbf{K}_0 \| \leq k^*_o, \quad k^*_o > 0 \), but direction of this vector within the BRF is arbitrary. Therefore, at time \( t = t_0 \) total AM vector related to the whole mechanical system \( \mathbf{G}_0 = \mathbf{K}_0 \) with \( \| \mathbf{G}_0 \| \leq g^o = k^*_o \). The inertial parameters of the RB are assumed to be known, the same for the possibility to measure vector \( \dot{\omega}(t) \) and vector \( \mathbf{H}(t) \). Let establish of a fixed unit vector \( \mathbf{f} = \mathbf{e}_y = \{0, 1, 0\} \) or \( \mathbf{f} = \mathbf{-e}_y = \{0, -1, 0\} \) is given within the BRF – the unit of a RB having the largest moment of inertia or the one opposite.

The problem consists in designing the GMC control law for achieving such condition of a gyrostat (3.1) with specified accuracy at any time moment \( t = T_f \).
\[ K_f = J \omega_t; \quad \omega_t = \omega_t f; \quad \mathcal{H}_f = \mathcal{H} f, \]  

where \( K_f \equiv K(T_f) \); \( \mathcal{H}_f \equiv \mathcal{H}(T_f) \); \( \omega_t \equiv \omega(T_f) \) and module \( \mathcal{H}_f \) of the total GMC AM's is established, in particular, as \( \mathcal{H}_f = 0 \). Taking into account the identity \( J_o \omega_t + \mathcal{H}_f = g_o \), one can find the obvious relation \( \omega_t = (g_o - \mathcal{H}_f)/J_o \).

After solving this vital problem, it is necessary to ensure the distribution of the AM \( \mathcal{H} \) and control torque \( M = M^w \) or \( M = M^s \) vectors between four RWs or GDs, accordingly. It is desirable to have the explicit distribution law (DL) allowing to obtain all movement characteristics for each electromechanical actuator based on the analytical relations. The GMC with collinear GD gimbal axes obtains a significant advantage: all its singular states are passable [26]. At 4 GDs the same approach is possible only for 2-SPE scheme, see Fig. 3. It is also necessary to consider a respinup of the flexible spacecraft structure through using four GDs.

5 Synthesis of main control law

An AM vector \( G(t) = J \omega(t) + \mathcal{H}(t) \) of the whole mechanical system with no external torques has its value unchanged within any inertial reference frame (IRF), in accordance with the theoretical mechanics principles. Unit vector \( g(t) \equiv \{g(t)\} = G(t)/g_o \) is also a fixed one within the IRF, but within the BRF this unit is moving in accordance with equation

\[ \dot{g}(t) = -\omega(t) \times g(t). \]  

Let us assume that at a time moment \( t_0 \) there is also calculated indicator \( a_t = \text{Sgn} \, C_q(t_0) \) of unit vector \( f = a_t e_y \). At notation \( \zeta(t) = g(t) - f \) as a nearby measure for the unit vectors \( g \) and \( f \), it is suitable to use scalar function

\[ v_p(t) \equiv v_p(\zeta(t)) = \zeta^2(t)/2 = 1 - \langle f, g(t) \rangle \gg 0. \]  

This function has positive values under \( g(t) \neq f \) and obtains zero value at the above vectors coincided only. With above selection of unit vector \( f = a_t \{0, 1, 0\} \), we always have \( v_p(t_0) \leq 1 \). Taking into account standard vector identities \( \langle a, (b \times c) \rangle = \langle b, (c \times a) \rangle = \langle c, (a \times b) \rangle \) and \( \zeta(t) \equiv -\omega(t) \times g(t) \) by (5.1), we have derivative of function \( v_p \) (5.3) as follows

\[ \dot{v}_p = \langle \zeta(t), \dot{\zeta}(t) \rangle = \langle \zeta(t), \eta(t) \rangle. \]  

Vectors \( \xi(t) \) and \( \zeta(t) \) are connected by identities
Taking into account that due to (5.2) \( \dot{\alpha} = \alpha \), Theorem.
For the RB movement required
where vector \( \mu \mu \) \( M \) respect to required equilibrium, where any constant
equation (3.1) is presented in simplest form
moreover vector \( \xi(t) \) is moving by equation
\[
\dot{\xi} = \eta - \Phi; \Phi \equiv \omega_0 \xi + g(t, \eta) + (\eta + \omega_0 f) \xi^2 / 2.
\] (6.5)

Taking into account that due to (5.2) \( \omega(t) = \eta(t) \) and the relations
\[
G(t) = g_0 g = g_0 f + g_0 (g(t) - f) = K_f + \mathcal{H}_f + g_0 \xi(t); \quad v \equiv J_\eta - g_0 \xi = -(\mathcal{H} - \mathcal{H}_f); \quad v = J_\omega + \omega \times G,
\] equation (3.1) is presented in simplest form
\[
v \equiv J_\eta - g_0 \xi = M = -\mathcal{H}.
\] (5.7)

Function \( v_e(v) \equiv v^2 / (2 j_h) = (\mathcal{H} - \mathcal{H}_f)^2 / (2 j_h) \) defines a GMC kinetic energy at its motion with respect to required equilibrium, where any constant \( j_h > 0 \) presents the inertia properties.

The RB movement required \( O_\eta \equiv \{ \xi = 0; \eta = 0 \} \) is the same \( O_v \equiv \{ \xi = 0; v = 0 \} \) due to identities (5.5). For notation \( \rho^2(t) \equiv \| \xi(t) \|^2 + \| \eta(t) \|^2 \) in the first let consider any small domain
\[
O \equiv \{ \| \xi \| < \epsilon_1 \} \cap \{ \| \rho \| < \epsilon_p = \text{const} \},
\]
within which no constrains for control torque \( M \) vector have occurred. To justify structure of the control torque \( M \) law into equation (5.7), we introduce Lyapunov function
\[
V = abv_p(\xi) + (a / j_h) \langle v, P_\xi \rangle + v_e(v),
\] (5.8)
where scalar parameters \( a > 0, b > 0 \) and \( P \) is a constant definitely-positive matrix. Taking into account that \( \xi^2 \equiv 2^2 / (1 + (1 - \xi^2)^{1/2}) \) due to identity (5.5) and well-known Schur lemma for a symmetric composite matrix, function \( V(\xi, \eta) \) is definitely positive with respect to any vector \( \xi \) and \( \eta \) into domain \( O \) for large value of parameter \( b \) and small value of parameter \( a \).

The derivative of this function with (5.4) and (5.7) taken into account has the form
\[
\dot{V} = ab \langle \xi, \eta \rangle + [\langle M, \mu \rangle + \langle v, P_\xi \rangle] / j_h,
\] (5.9)
where vector \( \mu \equiv v + a P_\xi \). For domain \( O \) the GMC control law is selected in the form
\[
M = M_\xi \equiv -q j_h D \mu = -m [\xi + k D v]
\] (5.10)
with parameters \( q > 0, m = q j_h a > 0, k = 1 / a > 0 \) and definitely-positive matrix \( D = \mathbf{P}^{-1} \).

**Theorem.** For the RB movement required \( O_\eta \) of the system’s model (5.6), (5.7) with control law (5.10) the property of exponential stability
\[
\rho(t) \leq \beta \rho(t_0) \exp(-\alpha(t - t_0)),
\] (5.11)
where \( \alpha, \beta = \text{const} > 0 \), is guaranteed for arbitrary vector of initial conditions \( \{ \xi(t_0), \eta(t_0) \} \in O_0 \subseteq O \) at chosen large value \( q(g_0) \).

**Proof.** Derivative (5.9) of function (5.8) by relation (5.6) taken into account is presented as follows
\[
\dot{V} = -q a^2 \langle \xi, P_\xi \rangle + a(b \langle \xi, \eta \rangle - 2q \langle \xi, J_\eta \rangle) - q \langle v, D v \rangle + (a / j_h) \langle v, P(\eta - \phi(\xi, \eta)) \rangle,
\] (5.12)
where vector \( v = J_\eta - g_0 \xi \) and function \( \phi(\cdot) \) was defined in (5.6).

Taking into account \( \langle v, D v \rangle = \langle D J_\eta, \eta \rangle - 2g_0 \langle D J_\eta, \xi \rangle + g_0^2 \langle D \xi, \xi \rangle \) and analogous representations of the terms \( \langle v, P_\eta \rangle, \langle v, P_\xi \rangle, \langle v, P \phi \rangle \) in (5.12), and also identities (5.5), one makes sure of the majoring
$V \leq -W(\xi, \eta)$, where scalar function $W(\xi, \eta)$ is definitely positive with respect to variables $\xi$ and $\eta$ for large values of parameters $b$ and $q$, depending on total AM value $g_0$, thanks to Schur lemma. By analogy with Ye.Ya. Smirnov [10] there is proved that $W(t) \to 0$ at $t \to \infty$ and function $V(t)$ is decreased monotonically. Standard estimates [12, 14] are derived from majoring functions $V$ and $W$ by quadratic forms $a_1 \rho^2 \leq V \leq a_2 \rho^2, a_1 > 0$; $b_1 \rho^2 \leq W \leq b_2 \rho^2, b_1 > 0$, from where condition (5.11) is appeared with the parameters $\alpha = b_1/(2a_2)$ and $\beta = (a_2/a_1)^{1/2}$.

Due to identity $v = J_{\eta} - g_o \zeta = -(H - H_f)$ control law (5.10) is appeared in very simple form

$$M_\xi(t) = -m(\xi(t) - kD[H(t) - H_f])$$

interior to nearest neighborhood of required gyrostat state $O_\eta$. Outside this neighborhood the control law is not effective because of various equilibrium manifolds [19] which exist at conditions

$$M_\xi = J_{\eta} - g_o \zeta \equiv 0; \quad J_{\eta} - g_o \zeta = c; \quad aP_\xi = -c$$

with a constant vector $c \neq 0$. Therefore other simple control laws are needed for fastest the SC respining without sticking its motion on any equilibrium manifold differing from the state $O_\eta$.

For notations $M_\xi(t) \equiv -m[e_\xi(t)SgnC_q(t) - kD[H(t) - H_f]]$ and $M'(t) \equiv -m^\epsilon \{Sgn g_\xi(t)\}$, where $m^\epsilon$ is a large constant parameter, developed control law has the form

$$M = \begin{cases} M_\xi(t) & \|\xi(t)\| \leq \epsilon_1; \\ M_\xi'(t) & \epsilon_1 \leq \|\xi(t)\| \leq \epsilon_2; \\ M'(t) & \|\xi(t)\| > \epsilon_2, \end{cases}$$

(5.13)

where for example, parameters $\epsilon_1 = 0.1$ (angle $\varphi = 6^\circ$) and $\epsilon_2 = 0.5$ (angle $\varphi = 30^\circ$).

6 The GMC distribution laws

For any GMC having an excessive structure the most vital control aspect is selection of a distributing law (DL) of required total GMC’s AM between electromechanical actuators. It is desirable to have an explicit DL based on analytical relations.

Unlike well-known RW DLs based on pseudo-inversion of matrix $A_r = J_r A_q$, fundamental idea of employed DL is in achieving the strict uniformity in terms of the saturation resources for RWs pairs. In normalized form the DL is described by relation

$$df_p(h)/dt = \Phi_p(f_p(h)) \equiv -Sat(\phi_p, \mu_p f_p(h)), \quad (6.1)$$

where function $f_p(h) = \bar{x}_1 - \bar{x}_2 + \rho(\bar{x}_1 \bar{x}_2 - 1)$; $\bar{x}_1 \equiv x_1/q_1; \bar{x}_2 \equiv x_2/q_2; q_1 = (4C_q^2 - s^2)^{1/2}, s = y, z$, and $\rho (0 < \rho < 1), \phi_p$ and $\mu_p$ are positive parameters. The GMC angular momentum is distributed as per condition $f_p(h) = 0$ firstly among the RW pairs as $q \equiv q_1 + q_2; b \equiv b/2; c = (q_1 - q_2)b + \rho(q_1 q_2 - b^2); \Delta \equiv (q/\rho)(1 - (1 - 4\rho c/q_1^2)^{1/2}); \quad x_1 = (x + \Delta)/2; \quad x_2 = (x - \Delta)/2,$

and then among two RWs in each pair. To define the column $M$, the relation $A_r h = \dot{H}$ is supplemented with the equation $(a_f(h), h) = \Phi_p(f_p(h))$, where $a_f(h) = \partial f_p(h)/\partial h$. As a result, we obtain four linear equations having positive determinant for all internal points within $S$ domain. Thus, the vector $M$ can be analytically calculated if vector $H$ is known.

For 2-SPE scheme normed AM vector $h(\beta) = A_q^{-1}H(\beta)/h_\beta$ is distributed between GDs by the DL

$$f_p(\beta) = (\bar{x}_1 - \bar{x}_2) + \rho(\bar{x}_1 \bar{x}_2 - 1) = 0; \quad \bar{x}_1 = x_{12}/q_y; \quad \bar{x}_2 = x_{34}/q_z; \quad q_\beta = (4 - s^2)^{1/2}, s = y, z.$$

For $\rho = 2\sqrt{6}/5$ this DL ensures global maximum of Grame determinant $G = det(A_h A^1_h)/h_\beta$ = 64/27 and maximum module of the warranted control torque vector $M = M^e$ (3.5) in an arbitrary direction.
for the "park" state \( h(\beta) = 0 \), as well as large singularityless central part inside of the GMC AM’s variation domain \( S = \{ x^2 + y^2 + z^2 - 2q_yq_z < 8; |y| < 2; |z| < 2 \} \) and only curves in the set of smoothly passed GMC internal singularities \( Q_{yz}(\beta) = Q_y^p \cup Q_z^p \), where

\[
Q_y^p = Q_y^s \cap S^y; S^y = \{ s = 0; |s_1| = |s_2| = 1 \}, s = y, z;
\]

\[
Q_z^p = \{ (x_{34}/(2p))^2 + (z/2)^2 = 1; x_{34} < 0 \}; Q_z^o = \{ (x_{12}/(2p))^2 + (y/2)^2 = 1; x_{12} > 0 \}.
\]

At "right-sided differential relay-hysteresis" tuning of the DL due to \( D + f_p(\beta) = \Phi_p(f_p(\beta), h(\beta)) \) with positive constants \( \phi_p, \mu_p \) and \( l_p \), where

\[
\Phi_p(\cdot) \triangleq \begin{cases}
-\text{Sat}(\phi_p, \mu_p f_p(\beta)) & h \in S \setminus Q_{yz}; \\
\phi_p \text{Relh}(a_s, l_p, r_s) & h \in Q_y^p, s = y, z;
\end{cases}
\]

\[
\text{Relh}(a, l_p, x) = (1, \text{if } x > -l_p) \vee (-1, \text{if } x < l_p), \quad \text{Relh}(a_s, l_p, r_s(\beta(t_0))) = a_s \in \{-1; 1\}, s = y, z;
\]

\[
r_y = M_\pi(\beta_1 - \beta_2 - \pi); \quad r_z = M_\pi(\beta_3 - \beta_4 - \pi); \quad M_\pi(\alpha) \equiv (\alpha, \text{if } |\alpha| \leq \pi) \vee (\alpha - 2\pi \text{Sign}(\alpha), \text{if } |\alpha| > \pi),
\]

this distribution law ensures its belonging to imaginary singular set \( Q_{yz}(\beta) \) only at separate time moments, and bijectively connects vector \( \mathbf{M}^g \) with vectors \( \beta \) and \( \dot{\beta} \).

### 7 Computer simulation

Based on above control laws the RB motion have been simulated with the following parameter values: \( J_x = 2900 \), \( J_y = 3600 \) and \( J_z = 870 \text{ km}^2 \) [27]. Fig. 4 summarizes the simulation results for initial
position of the SC AM vector $G(t_0)$ with module $g_0 = 300$ Nms along unit $g(t_0) = \{0, 0, 1\}$ within the BRF and its final position coincided with unit $f = \{0, 1, 0\}$. For clearness here the simplest canonical GMC schemes were applied:
Nonlinear Dynamics of Mini-satellite Respinup by Weak Internal Control Torques

- canonical scheme on 3 RWs with constrains $m^m = 0.15 \text{ Nm}$ and $h^m = 5 \text{ Nms}$;
- the 2-SPE scheme on 4 GDs with angle $\gamma_g = \pi/4$, $h_g = 7.5 \text{ Nms}$ and constrain $u^m = 10 \text{ deg/s}$.

Some results on the flexible spacecraft dynamics during its respinup by four GDs with the same parameters, are presented in Fig. 5.

8 Recent research

Optimization [28] and robust gyromoment control problems [29, 30] were also considered for respinup of the flexible spacecraft. In addition to [31, 32] problems of the SAPs guidance on the Sun were studied. Moreover the SC inertia tensor is changed into the BRF and the GMC’s control torque vector $M = M^\theta$ is re-calculated for principal central axes for variable SC inertia tensor.

Conclusions

We have presented principal aspects only on nonlinear dynamics related to the controlled coincidence of any SC body axis with the SC AM vector by the RWs or the GDs. Method for synthesis of nonlinear control law and analytical proof of stability for the required spacecraft rotation mode were developed.

Obtained results were verified by the careful computer simulation of strongly nonlinear processes.

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References


