Traveling-wave solutions of the generalized Zakharov equation with time-space fractional derivatives

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Abstract. In this paper, we implement a Functional Variable Method (FVM) for extracting various kinds of exact traveling wave solutions of the time-space fractional-order generalized Zakharov equation (FGZE). The method is extremely simple and effective for handling nonlinear equations with fractional derivatives arising in mathematical physics.

1 Introduction

Nonlinear partial differential equations are very important in various disciplines, such as physics, mechanics, biology, chemistry, etc. Recently many new approaches have been proposed to find exact solutions of nonlinear PDEs, for example the improved F-expansion method [10], the projective Riccati equation method [5], the Jacobi elliptic function expansion method [16] and the tanh-function method [17]. For integrable nonlinear differential equations, we find the Hirota method [7], the Backlund transform method [8] and the Exp-function method [15]. More recently, Zerarka et al. [18] proposed direct and concise method called the functional variable method (FVM) for solving nonlinear evolution equations to find new types of solution. On the other hand, fractional differential equations (FDEs) have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, biology and other areas [11][14]. For solving (FDEs) several methods have also been provided for more details see [2][3][5][6] and the references therein.

The main objective of the present paper is to show the effectiveness of a fractional version of the functional variable method to obtain many exact traveling waves to the fractional generalized Zakharov equation (FGZE). The solution procedure has been implemented in Mathematica.7 software.

The paper is arranged as follows, in Section 2, we give some basic definitions about the fractional derivative used in this work and we describe briefly the (FVM). In Section 3, we apply the method

2010 Mathematics Subject Classification: 35C08, 35R11, 97M50.

Keywords: Exact travelling wave, Functional-variable method, Jumarie’s fractional derivative, Generalized Zakharov equation.
to find various type of exact solutions to the (FGZE). Further 2D- and 3D- plots for the obtained solutions are given. Concluding remarks are listed in Section 4.

2 Preliminaries

In this section some necessary mathematical background is presented.

2.1 The Jumarie’s Riemann-Liouville derivative (JMRLD)

There are different definitions for fractional derivatives for more details see [14] and the references therein. In this paper we use the modified Riemann-Liouville derivative defined by Jumarie [9]. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) denote a continuous (but not necessarily differentiable) function. the (JMRLD) is defined by

\[
D^\alpha_t f(t) = \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} (f(s) - f(0)) \, ds, & \text{for } 0 < \alpha < 1, \\
\left[f^{(\alpha-n)}(t) \right]^{(n)} , & \text{for } 1 \leq n \leq \alpha < n+1.
\end{cases}
\]

We list here some important properties for the (JMRLD) as follows

\[
D^\alpha_t t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha},
\]

(2.1)

\[
D^\alpha_t (f(t)g(t)) = g(t)D^\alpha_t f(t) + f(t)D^\alpha_t g(t),
\]

(2.2)

\[
D^\alpha_t [g(t)] = f'_g[g(t)]D^\alpha_t g(t) = D^\alpha_g f[g(t)](g'(t))^\alpha.
\]

(2.3)

2.2 Description of the (FVM)

In [18], Zerarka et al proposed the so-called functional variable method to solve a wide class of linear and nonlinear wave equations. Right after this pioneer work, this method became popular among the research community, and many studies refining the initial idea for finding exact solutions of some real and complex nonlinear evolution equations have been published (see [4] and the references therein). The advantage of (FVM) is that one treats nonlinear problems by essentially linear methods, based on which it is easy to construct in full the exact solutions such as soliton-like waves, compacton and non-compacton solutions, trigonometric function solutions, pattern soliton solutions, black solitons or kink solutions, and so on.

We now describe the (FVM) for finding explicit solutions to the time-space fractional differential equations. Let us consider a time-space fractional differential equation with independent variables \((x, y, z, t, \ldots)\) and a dependent variable \(u\):

\[
P(u, D^\alpha_t u, D^\alpha_x u, D^\alpha_{xx} u, D^\alpha_{tx} u, D^2_{tt} u, \ldots) = 0,
\]

(2.4)

where the subscript denotes partial derivative. Using the following variable transformations

\[
\left\{ \begin{array}{l}
\xi = \frac{x^\alpha}{\Gamma(\alpha + 1)} + \frac{y^\alpha}{\Gamma(\alpha + 1)} + \frac{z^\alpha}{\Gamma(\alpha + 1)} + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \ldots
\end{array} \right.
\]

(2.5)
where $\alpha$ is the order of fractional derivation and $\nu$ is the wave speed (to be determined later); the fractional partial differential equation $P$ is converted to an ordinary fractional differential equation (FODE):

$$Q(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, \ldots) = 0. \tag{2.6}$$

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_\xi = F(U), \tag{2.7}$$

and some successive derivatives

$$\begin{align*}
U_{\xi\xi} &= \frac{1}{2}(F(U)^2)' \\
U_{\xi\xi\xi} &= \frac{1}{2}(F(U)^2)'' \sqrt{F(U)^2}, \\
U_{\xi\xi\xi\xi} &= \frac{1}{2} \left[(F(U))'' \sqrt{F(U)^2}\right], \\
&\vdots
\end{align*} \tag{2.8}$$

where \(\cdot'\) denotes differentiation with respect to $U$. Substitution of (2.7) and (2.8) into (2.6) reduces it to an ordinary differential equation in terms of $U$, $F$ and its derivatives as

$$R(U, F, F', F'', \ldots) = 0. \tag{2.9}$$

After integration, the Equation (2.9) provides the expression of $F$, and this in turn together with the equations (2.7) and (2.6) give the appropriate solutions of the original equation (2.4).

3 Explicit exact traveling waves to the FGZE

There are various vector valued coupled nonlinear equations that describe many physical phenomena. One of them is the Generalized Zakharov Equation (GZE) [12], which is a realistic model for plasma, and can be written as:

$$\begin{align*}
iq_t + aq_{xx} + b|q|^2 q &= qr, \\
r_{tt} + k^2 r_{xx} &= (|q|^2)_{xx},
\end{align*} \tag{3.1}$$

where $q(x,t)$ is the envelope of the high-frequency electric field, and $r(x,t)$ is the plasma density measured from its equilibrium value, the independent variables $x$ and $t$ are the spatial and temporal variables respectively, while $a$ is the group velocity dispersion and $b$ represents the power law nonlinearity. Obviously, the (GZE) is a strongly nonlinear system; therefore, it is a difficult task to obtain its solitary wave solutions.

In this paper, we aim to construct explicit exact solutions to the (FGZE), which is in the following form

$$\begin{align*}
iD_\alpha^\alpha q + aD_\alpha^\alpha q + b|q|^2 q &= qr, \\
D_\alpha^\alpha r + k^2 D_\alpha^\alpha r &= D_\alpha^\alpha (|q|^2),
\end{align*} \tag{3.2}$$
In order to solve (3.2) by the aid of traveling wave hypothesis, we assume that solution is in the following frame:

\[
\begin{align*}
q(x,t) &= e^{i\phi(x,t)} U(\xi), \\
r(x,t) &= V(\xi), \\
\phi(x,t) &= -\kappa \frac{x}{\Gamma(\alpha+1)} + \omega \frac{t}{\Gamma(\alpha+1)} + \theta, \\
\xi &= \frac{x}{\Gamma(\alpha+1)} - \frac{\nu}{\Gamma(\alpha+1)} t.
\end{align*}
\] (3.3)

where \(U(\xi)\) represents the traveling wave profile, \(\nu\) is the speed of the wave, \(\kappa\) is the frequency while \(\omega\) is the wave number and \(\theta\) is the phase constant. Substituting (3.3) into (3.2) yields

\[
\begin{align*}
(\nu^2 - 1) V_{\xi\xi} &= (U^2)_{\xi\xi}, \\
V &= \frac{U^2}{\nu^2 - 1}.
\end{align*}
\] (3.4)

Integrating (3.4) twice, taking the integration constant as zero, since the search is for a soliton solution, and decomposing into real and imaginary parts, we get the following pair of equations

\[
\begin{align*}
U'' - (\omega + \kappa^2) U + \left(\frac{1}{\nu^2 - 1} - b\right) U^3 &= 0, \\
\nu &= -2\kappa.
\end{align*}
\] (3.5)

where \(a\) is assumed to be 1.

### 3.1 Exact solutions by the (FVM)

Substituting (2.7) and (2.8) into (3.5) we obtain

\[
(F(U)^2)' = 2 \left[(\omega + \kappa^2) U + \left(\frac{1}{\nu^2 - 1} - b\right) U^3\right].
\] (3.6)

Integrating (3.6) with respect to \(U\), we have

\[
F^2(U) = (\omega + \kappa^2) U^2 + \frac{1}{2} \left(\frac{1}{\nu^2 - 1} - b\right) U^4.
\] (3.7)

Then, four cases arise:

- **Case 1**: If \(\omega + \kappa^2 > 0\) and \(\frac{1}{\nu^2 - 1} - b > 0\), we have

\[
F(U) = U \sqrt{\frac{2(\omega + \kappa^2)}{\nu^2 - 1} - b} + U^2 \sqrt{\frac{1}{2} \left(\frac{1}{\nu^2 - 1} - b\right)},
\] (3.8)

Then we get from (2.7) and (3.7) that

\[
\frac{dU}{U \sqrt{\frac{2(\omega + \kappa^2)}{\nu^2 - 1} - b} + U^2} = \pm \sqrt{\frac{1}{2} \left(\frac{1}{\nu^2 - 1} - b\right)},
\] (3.9)

By integration, we get
\[
\begin{align*}
U(\xi) &= \pm \sqrt{\frac{2 \kappa^2 + 2\omega}{(\nu^2 - 1)^{-1}} - b} \cosh \left( \sqrt{\kappa^2 + \omega\xi} \right) \\
V(\xi) &= \frac{(2 \kappa^2 + 2\omega) \left( \cosh \left( \sqrt{\kappa^2 + \omega\xi} \right) \right)^2}{(\nu^2 - 1) \left( (\nu^2 - 1)^{-1} - b \right)}.
\end{align*}
\] (3.10)

Coming back to (3.3) we find the exact solitary waves:

\[
\begin{align*}
q(x,t) &= \pm e^{\left( -\frac{\nu^\alpha t}{\Gamma(\alpha + 1)} - \frac{\nu^\alpha t}{\Gamma(\alpha + 1)} \right)} \sqrt{\frac{2 \kappa^2 + 2\omega}{(\nu^2 - 1)^{-1}} - b} \cosh \left( \sqrt{\kappa^2 + \omega \left( x^{\alpha} - \frac{\nu^\alpha t}{\Gamma(\alpha + 1)} \right)} \right), \\
r(x,t) &= \frac{(2 \kappa^2 + 2\omega) \left( \cosh \left( \sqrt{\kappa^2 + \omega \left( x^{\alpha} - \frac{\nu^\alpha t}{\Gamma(\alpha + 1)} \right)} \right) \right)^2}{(\nu^2 - 1) \left( (\nu^2 - 1)^{-1} - b \right)}.
\end{align*}
\] (3.11)

Fig. 1 Soliton solutions (3.11): (a) modulus of \( q(x,t) \), (b) \( r(x,t) \) for \( \kappa = \sqrt{\frac{3}{3}} \), \( \omega = 0.5 \), \( \nu = -\frac{2\sqrt{3}}{3} \), \( \alpha = 0.7 \) and \( b = 1 \).

Fig. 2 2D plots of the soliton solutions (3.11) at different times: (a) modulus of \( q(x,t) \) and (b) \( r(x,t) \).
\begin{itemize}
  \item Case 2: If $\omega + \kappa^2 > 0$ and \( \frac{1}{\sqrt{v^2 - 1}} - b < 0 \), we get from (2.7) and (3.7) that
  \[
  \frac{dU}{U \sqrt{a^2 + U^2}} = \pm \sqrt{\frac{1}{2} \left( b - \frac{1}{\sqrt{v^2 - 1}} \right)} dU \quad \text{with} \quad a = \sqrt{\frac{2(\omega + \kappa^2)}{(b - \frac{1}{\sqrt{v^2 - 1}})}}. \tag{3.12}
  \]

  By integration with respect to $U$ we get
  \[
  \begin{align*}
  U(\xi) &= \sqrt{-\frac{2 \kappa^2 - 2 \omega}{(v^2 - 1)^{-1} - b}} \operatorname{sech} \left( \sqrt{\frac{\kappa^2 + \omega}{\xi} \Gamma(\alpha + 1)} \right), \\
  V(\xi) &= \frac{(-2 \kappa^2 - 2 \omega) \left( \operatorname{sech} \left( \sqrt{\frac{\kappa^2 + \omega}{\xi} \Gamma(\alpha + 1)} \right) \right)^2}{(v^2 - 1) \left( (v^2 - 1)^{-1} - b \right)}. \tag{3.13}
  \end{align*}
  \]
  \end{itemize}

  From which we deduce the exact solitary waves
  \[
  \begin{align*}
  q(x, t) &= \pm e^{-i \left( \frac{\kappa^2 + \omega}{\Gamma(\alpha + 1)} \right)^{\alpha} \Gamma(\alpha + 1)} \sqrt{2} \sqrt{\frac{(k^2 + \omega)(v - 1)(v + 1)}{bv^2 - b - 1}} \operatorname{sech} \left( \sqrt{\frac{k^2 + \omega}{\Gamma(\alpha + 1)} \Gamma(\alpha + 1)} \right), \\
  r(x, t) &= \frac{(-2 \kappa^2 - 2 \omega) \left( \operatorname{sech} \left( \sqrt{\frac{\kappa^2 + \omega}{\Gamma(\alpha + 1)} \Gamma(\alpha + 1)} \right) \right)^2}{(v^2 - 1) \left( (v^2 - 1)^{-1} - b \right)}. \tag{3.14}
  \end{align*}
  \]

\begin{figure}
\centering
\includegraphics[width=0.45\textwidth]{fig3a.png}
\includegraphics[width=0.45\textwidth]{fig3b.png}
\caption{Soliton solutions (3.14): (a) modulus of $q(x, t)$, (b) $r(x, t)$ for $\kappa = 2$, $\omega = -1$, $v = -4$, $\alpha = 0.9$ and $b = 1$.}
\end{figure}
• Case 3: If \( \omega + \kappa^2 < 0 \) and \( \frac{1}{\nu^2 - 1} - b > 0 \), we get from (2.7) and (3.7) that

\[
F(U) = \pm U \sqrt{-a^2 + U^2} \sqrt{\frac{1}{2} \left( b - \frac{1}{\nu^2 - 1} \right)} \quad \text{with} \quad a = \sqrt{-\frac{2(\omega + \kappa^2)}{\sqrt{\nu^2 - 1} - b}}.
\]

Then

\[
\int \frac{dU}{U \sqrt{U^2 - a^2}} = \pm \sqrt{\frac{1}{2} \left( \frac{1}{\nu^2 - 1} - \nu \right)} d\xi,
\]

which yields

\[
\ln \left( \frac{a - \sqrt{U^2 - a^2}}{|U|} \right) = \pm a \sqrt{\frac{1}{2} \left( \frac{1}{\nu^2 - 1} - \nu \right)} \xi + \xi_0,
\]

Consequently

\[
\begin{align*}
U(\xi) &= \pm \sqrt{-(\kappa^2 + \omega) \left(-1/2 (\nu^2 - 1)^{-1} + 1/2b\right)^{-1}} \sec \left( \sqrt{-\kappa^2 - \omega \xi} \right), \\
V(\xi) &= \frac{(\kappa^2 + \omega) \left( \sec \left( \sqrt{-\kappa^2 - \omega \xi} \right) \right)^2}{(\nu^2 - 1) \left(-1/2 (\nu^2 - 1)^{-1} + 1/2b\right)}.
\end{align*}
\]

and then, the solitary waves are

\[
\begin{align*}
q(x,t) &= \pm e^{i \left( -\frac{x^\alpha}{\Gamma(\alpha + 1)} + \frac{2w + \kappa^2}{\nu^\alpha (\alpha + 1)} \right)} \sqrt{-\frac{2(w + \kappa^2)}{\nu^2 - b}} \sec \left( \sqrt{-(w + \kappa^2)\xi} \left( \frac{x^\alpha}{\Gamma(\alpha + 1)} - \frac{\nu^\alpha}{\Gamma(\alpha + 1)} \right) \right), \\
r(x,t) &= \frac{(\kappa^2 + \omega) \left( \sec \left( \sqrt{-\kappa^2 - \omega \left( \frac{x^\alpha}{\Gamma(\alpha + 1)} - \frac{\nu^\alpha}{\Gamma(\alpha + 1)} \right) \right)^2}{(\nu^2 - 1) \left(-1/2 (\nu^2 - 1)^{-1} + 1/2b\right)}.
\end{align*}
\]
Fig. 5 Soliton solutions (3.19): (a) modulus of $q(x,t)$, (b) $r(x,t)$ for $\kappa = 1$, $\omega = 0.5$, $\nu = -2\kappa$, $\alpha = 0.7$ and $b = 1$.

Fig. 6 2D plot of the soliton solutions (3.19) at different times: (a) modulus of $q(x,t)$ and (b) $r(x,t)$

- Case 4: If $\omega + \kappa^2 < 0$ and $\frac{1}{\nu^2 - 1} - b < 0$, we get from (2.7) and (3.7) that

$$F(U) = \pm iU \sqrt{\frac{2(\omega + \kappa^2)}{\frac{1}{\nu^2 - 1} - b}} + U^2 \sqrt{\frac{1}{2} \left(b - \frac{1}{\nu^2 - 1}\right)}.$$  \hspace{1cm} (3.20)

In an analogous way as in the previous cases, we get

$$\frac{1}{\sqrt{\frac{2(\omega + \kappa^2)}{\frac{1}{\nu^2 - 1} - b}}} \text{arccsch} \left( \frac{|U|}{\sqrt{\frac{2(\omega + \kappa^2)}{\frac{1}{\nu^2 - 1} - b}}} \right) = \pm i \sqrt{\frac{1}{2} \left(b - \frac{1}{\nu^2 - 1}\right)} \xi + \xi_0.$$ \hspace{1cm} (3.21)

and then

$$U(\xi) = \pm \sqrt{(-\kappa^2 - \omega) \left(\frac{1}{2} (\nu^2 - 1)^{-1} - 1/2b\right)^{-1}} \csc \left(\sqrt{-\kappa^2 - \omega}\xi\right),$$

$$V(\xi) = \frac{(-\kappa^2 - \omega) \left(\csc \left(\sqrt{-\kappa^2 - \omega}\xi\right)\right)^2}{\left(\frac{1}{2} (\nu^2 - 1)^{-1} - 1/2b\right) (\nu^2 - 1)}.$$ \hspace{1cm} (3.22)
Thus we obtain the solitary waves

\[
\begin{align*}
q(x,t) &= \pm e^{i\left(-\frac{\kappa\alpha}{\Gamma(\alpha+1)} + \frac{\nu\alpha}{\Gamma(\alpha+1)}\right)} \sqrt{-\kappa^2 - \omega} \csc \left(\sqrt{-\kappa^2 - \omega} \left(\frac{x\alpha}{\Gamma(\alpha+1)} - \frac{\nu t\alpha}{\Gamma(\alpha+1)}\right)\right), \\
r(x,t) &= \frac{(-\kappa^2 - \omega) \left(\csc \left(\sqrt{-\kappa^2 - \omega} \left(\frac{x\alpha}{\Gamma(\alpha+1)} - \frac{\nu t\alpha}{\Gamma(\alpha+1)}\right)\right)\right)^2}{\left(1/2 (\nu^2 - 1)^{-1} - 1/2b\right) (\nu^2 - 1)}.
\end{align*}
\]

(3.23)

![Fig. 7](image1.png)

Fig. 7 Soliton solutions (3.23): (a) modulus of $q(x,t)$, (b) $r(x,t)$ for $\kappa = 1$, $\nu = \omega = -2$, $\alpha = 0.8$ and $b = 1$.

![Fig. 8](image2.png)

Fig. 8 2D plot of the soliton solutions (3.23) at different times: (a) modulus of $q(x,t)$ and (b) $r(x,t)$.

4 Conclusion

We have successfully obtained exact traveling waves to the (FGZE) by using a fractional version of the (FVM). The procedure is extremely simple in its principle and quite easy to use, and it can be extended to other types of nonlinear fractional differential equations.
References