Waveform relaxation method for differential equations with fractional-order derivative

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Abstract. In this paper, we present a numerical computational approach for solving Caputo type fractional differential equations. This method is based on approximation of Caputo derivative in terms of integer order derivatives and waveform relaxation method. The utility of the method is shown by applying it to several examples. A comparative study indicates that our approach is more efficient and accurate than the ones available in the literature.

1 Introduction

Fractional calculus has recently evolved an interesting and important field of research due to its extensive applications in physical and technical sciences [1, 2]. A variety of results ranging from theoretical analysis to asymptotic behavior and numerical methods for fractional differential equations can be found in the literature on the topic [3, 4, 5, 6, 7, 8, 9].


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Some convergent splittings are given. Unfortunately, classical waveform relaxation is hampered by slow convergence, but this can be addressed by better transmission conditions, which gives rise to a new class of optimized waveform relaxation methods. In these methods, both voltage and current information is exchanged in a combination which can be optimized for the performance of the method [13].

In [14], some new numerical approaches based on piecewise interpolation for fractional calculus, and some new improved approaches based on the Simpson method for the fractional differential equations are proposed.

In this paper, we use waveform relaxation method in combination with the higher order algorithm for the fractional differential equations and pose the novel algorithms for a better convergence. A set of MATLAB routines for the implementation of the method as well as sample code used to solve the examples have been developed.

The structure of this paper is as follows. In Section 2, some definitions of fractional calculus are recalled, and the waveform relaxation method for fractional differential equations is illustrated in section 3. In Section 4, the numerical waveform relaxation methods for linear and nonlinear fractional differential equations are presented. We solve some examples in section 4. Section 5 contains conclusions.

2 Preliminaries

Let us recall some basic concepts of fractional calculus ([1, 15]).

**Definition 2.1.** The Riemann-Liouville fractional integral of order \( \alpha > 0 \) for a continuous function \( x \) is defined as

\[
J^\alpha_a x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \quad t > 0,
\]

\[
J^0_a x(t) = x(t).
\]  

**Definition 2.2.** The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) for a continuous function \( x \) is defined as

\[
RLD^\alpha_a x(t) = \frac{d^m}{dt^m} J^{m-\alpha}_a x(t), \quad m-1 < \alpha \leq m, m \in \mathbb{N}.
\]

**Definition 2.3.** The Caputo fractional derivative of order \( \alpha > 0 \) for a function \( x \in C^{m-1} \) with \( m \in \mathbb{N} \cup \{0\} \) is defined as

\[
D^\alpha_a x(t) = \begin{cases} 
J^{m-\alpha}_a x^{(m)}(t), & m-1 < \alpha < m, \\
\frac{d^m x(t)}{dt^m}, & \alpha = m.
\end{cases}
\]

**Definition 2.4.** A two-parameter Mittag-Leffler function is defined by the following series

\[
E^q_{\alpha, \beta}(t) = \sum_{k=0}^{\infty} \frac{(q)_k t^k}{\Gamma(\alpha k + \beta)}.
\]

where \((q)_0 = 1\) and \((q)_k = q(q+1)(q+2) \cdots (q+k-1), k = 1, 2, \cdots\).
3 Waveform relaxation method for fractional differential equations

For a mathematical description of waveform relaxation method, we consider the following initial value problem (IVP):

\[ \dot{c}D_{0}^{\alpha}x(t) = f(t, x(t)), \quad x(0) = x_{0}, 0 < \alpha \leq 1, t \in [0, T], \]  \tag{3.1} 

where \( f : [0, T] \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) is a given (linear or nonlinear) function, \( x_{0} \in \mathbb{R}^{n} \) and \( x(t) \) is an unknown vector.

In [12], a waveform relaxation algorithm for (3.1) is

\[
\begin{align*}
(x_{k+1}(t)) & = F(t, x^{(k)}(t), x^{(k+1)}(t)), t \in [0, T] \\
x^{(k+1)}(0) & = x_{0}, \quad k = 0, 1, \ldots,
\end{align*}
\]  \tag{3.2} 

where \( x^{(0)}(t) \) is a given initial function (for example, we can choose \( x^{(0)}(t) \equiv x_{0}, t \in [0, T] \) and the splitting function \( F : [0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) as \( F(t, x(t), x(t)) = f(t, x(t)) \) for any \( t \in [0, T] \) and \( x(t) \in \mathbb{R}^{n} \)). Here, at every waveform iteration \( k \), each equation in the system is solved for its corresponding component of \( x \) by using previous values of the other components as input. The structure of this process is analogous to that of Gauss-Jacobi relaxation method for solving linear systems of equations [16] and hence it is known as Jacobi waveform relaxation method. Here the solution \( x^{(k+1)}(t) \) of (3.2) is called the waveform relaxation solution of the original system. In fact the solution \( x^{(k+1)}(t) \) is an approximate analytic solution of (3.1).

For example, consider a fractional order Lorenz system [17]

\[
\begin{align*}
\dot{c}D_{0,t}^{0.995}x_{1}(t) & = 10(x_{2}(t) - x_{1}(t)), \\
\dot{c}D_{0,t}^{0.995}x_{2}(t) & = x_{1}(t)(28 - x_{3}(t)) - x_{2}(t), \\
\dot{c}D_{0,t}^{0.995}x_{3}(t) & = x_{1}(t)x_{2}(t) - \frac{8}{3}x_{3}(t), \\
x_{1}(0) & = 0.1, \quad x_{2}(0) = 0.1, \quad x_{3}(0) = 0.1, \quad t \in [0, T].
\end{align*}
\]  \tag{3.3} 

By using waveform relaxation method and the Gauss-Seidel splitting for (3.3), we obtain

\[
\begin{align*}
\dot{c}D_{0,t}^{0.995}x_{1}^{(k+1)}(t) & = 10(x_{2}^{(k)}(t) - x_{1}^{(k+1)}(t)), \\
\dot{c}D_{0,t}^{0.995}x_{2}^{(k+1)}(t) & = x_{1}^{(k+1)}(t)(28 - x_{3}^{(k+1)}(t)) - x_{2}^{(k+1)}(t), \\
\dot{c}D_{0,t}^{0.995}x_{3}^{(k+1)}(t) & = x_{1}^{(k+1)}(t)x_{2}^{(k+1)}(t) - \frac{8}{3}x_{3}^{(k+1)}(t), \\
x_{1}^{(k+1)}(0) & = 0.1, \quad x_{2}^{(k+1)}(0) = 0.1, \quad x_{3}^{(k+1)}(0) = 0.1, \quad t \in [0, T].
\end{align*}
\] 

Now we state the convergence criterion for problem algorithm (3.2).

**Theorem 3.1.** ([17]) Assume that the splitting function \( F : [0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) is continuous and satisfies the Lipschitz condition:

\( \| F(t, x_{1}(t), y_{1}(t)) - F(t, x_{2}(t), y_{2}(t)) \| \leq L_{1}\| x_{1}(t) - x_{2}(t) \| + L_{2}\| y_{1}(t) - y_{2}(t) \|, \)

for any \( (x_{1}(t), y_{1}(t)), (x_{2}(t), y_{2}(t)) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \), where \( L_{1} \) and \( L_{2} \) are positive constants.

Then, the waveform relaxation solutions obtained via (3.2) converge to the unique solution of (3.1).
The numerical representation for Caputo derivative of
\[ D_t^\alpha x(t) = \theta(\alpha, h) \sum_{i=0}^{m} W_{in} x^{(m)}(t_i) + O(h^2), \]
where \( \theta(\alpha, h) = \frac{h^{m-\alpha}}{\Gamma(m-\alpha+1)} \) and
\[ W_{in} = \begin{cases} n^{m-\alpha}(m-\alpha+1-n) + (n-1)^{m-\alpha+1} & i = 0, \\ (n-i-1)^{m-\alpha+1} + (n-i+1)^{m-\alpha+1} - 2(n-i)^{m-\alpha+1} & i = 1, \ldots, n-1, \\ 1 & i = n. \end{cases} \]

Replacing \( t \) by \( t_n \) in (3.2), we get
\[
\begin{cases}
(D_{0,t_n}^\alpha x^{(k+1)})(t_n) = F(t_n, x^{(k)}(t_n), x^{(k+1)}(t_n)), \\
x^{(k+1)}(0) = x_0, \quad k = 0, 1, \ldots.
\end{cases}
\]

Note that \( m = 1 \) as \( 0 < \alpha \leq 1 \). Approximating the left hand side of (4.2) by (4.1) yields
\[
\begin{cases}
\theta(\alpha, h) \sum_{i=0}^{n} W_{in} x^{(k+1)}(t_i) = F(t_n, x^{(k)}(t_n), x^{(k+1)}(t_n)), \\
x^{(k+1)}(0) = x_0, \quad k = 0, 1, \ldots.
\end{cases}
\]

Using the following approximation for \( (x')^{(k+1)}(t_i), i = 0, 1, \ldots, n \), in (4.3), we obtain
\[
\begin{pmatrix}
(x'(t_0) \\
(x'(t_1) \\
(x'(t_{n-1}) \\
(x'(t_n))
\end{pmatrix} = \frac{1}{2h} \begin{pmatrix}
-3 & 4 & -1 & \circ \\
-1 & 0 & 1 & \circ \\
\ldots & \ldots & \ldots & \circ \\
\circ & 1 & -4 & 3
\end{pmatrix} \begin{pmatrix}
(x'(t_0) \\
(x'(t_1) \\
(x'(t_{n-1}) \\
(x'(t_n))
\end{pmatrix} + O(h^2 \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}.
\]
Consider the following fractional differential equation \[ (4.4) \]
which clearly shows that our results are more accurate than the ones obtained in \[ 14 \].

5 Numerical examples

For the special case \( t = 0 \) by our method (OM) and by Li and Ye (LY) \[ 14 \] at

\[ (4.5) \]

where

\[
\begin{align*}
V_{0,n} &= \frac{1}{2h} (-3W_{0,n} - W_{1,n}), \\
V_{1,n} &= \frac{1}{2h} (4W_{0,n} - W_{2,n}), \\
V_{2,n} &= \frac{1}{2h} (-W_{0,n} + W_{1,n} - W_{3,n}), \\
V_{i,n} &= \frac{1}{2h} (W_{i-1,n} - W_{i+1,n}), \quad i = 3, 4, \cdots, n-3, \\
V_{n-2,n} &= \frac{1}{2h} (-W_{n-1,n} + W_{n-3,n} + W_{n,n}), \\
V_{n-1,n} &= \frac{1}{2h} (W_{n-2,n} - 4W_{n,n}), \\
V_{n,n} &= \frac{1}{2h} (3W_{n,n} + W_{n-1,n}).
\end{align*}
\]

We choose \( x^{(M)}(t) \) as an approximation to \( x(t) \) where \( M \in \mathbb{N} \). As in \[ 12 \], we consider \( x^{(0)}(t_n) = x_0 \). For the special case \( n = 1 \) in \( (4.3) \), we have

\[ (4.6) \]

where the following approximation \[ 18 \] for \( (x')^{(k+1)}(t_0) \) and \( (x')^{(k+1)}(t_1) \) has been used:

\[
(x')^{(k+1)}(t_0) = \frac{x^{(k+1)}(t_1) - x^{(k+1)}(t_0)}{h} + O(h) = (x')^{(k+1)}(t_1) = \Psi_{k+1}(h).
\]

By the above method we compute \( X = [X(t_1), \ldots, X(t_n)] = \Psi(h) \) and combining it with Richardson extrapolation

\[
X = \frac{4\Psi(\frac{h}{2}) - \Psi(h)}{3}. \tag{4.6}
\]

5 Numerical examples

Here we demonstrate the application of numerical schemes developed for fractional differential equations in the last Section. To show the efficiency of our schemes, we compare our results with the ones obtained in \[ 14 \] and \[ 19 \]. In all runs we put \( M = 5 \).

Example 5.1. Consider the following fractional differential equation \[ 14 \]:

\[
cD_{0}^{0.5} x(t) = \frac{2}{\Gamma(2.5)} t^{1.5} - x(t) + t^2, \quad x(0) = 0.
\]

The exact solution is \( x(t) = t^2 \). In Table 1, a comparison of the absolute errors in the solution obtained by our method (OM) and by Li and Ye (LY) \[ 14 \] at \( t = 10 \) for \( h = 0.1, 0.05, 0.025, 0.0125 \) is presented, which clearly shows that our results are more accurate than the ones obtained in \[ 14 \].
As a second example, let us consider the fractional differential equation given by

\[ t^{\alpha}x''(t) + \frac{2}{\Gamma(3 - \alpha)}t^{2 - \alpha} - \frac{1}{\Gamma(2 - \alpha)}t^{1 - \alpha} - x(t) + r^2 - t, \quad 0 < \alpha \leq 1. \]

The exact solution for \( \alpha = \frac{1}{2} \) is \( x(t) = t^2 - t \). In table 2-4, we list the absolute errors in the solution using OM and compare our results with the ones given LY in [14] at \( t = 10 \) for various values of \( h \) and \( \alpha = 0.7, 0.5 \) and 0.3 respectively. Clearly our results improve the ones obtained by LY.

**Example 5.2.** As a second example, let us consider the fractional differential equation given by

\[ t D_{0+}^{\alpha}x(t) = \frac{2}{\Gamma(3 - \alpha)}t^{2 - \alpha} - \frac{1}{\Gamma(2 - \alpha)}t^{1 - \alpha} - x(t) + t^2 - t, \quad 0 < \alpha \leq 1. \]

The exact solution for \( \alpha = \frac{1}{2} \) is \( x(t) = t^2 - t \). In table 2-4, we list the absolute errors in the solution using OM and compare our results with the ones given LY in [14] at \( t = 10 \) for various values of \( h \) and \( \alpha = 0.7, 0.5 \) and 0.3 respectively. Clearly our results improve the ones obtained by LY.

**Table 1.** Comparison of the numerical solutions obtained by the present method and those obtained by LY at \( t = 10 \) for various values of \( h \).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0361</td>
<td>0.0032</td>
<td>( 3 \times 10^{-6} )</td>
</tr>
<tr>
<td>0.05</td>
<td>0.017</td>
<td>6.7576 \times 10^{-4}</td>
<td>( 6 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.025</td>
<td>0.0039</td>
<td>1.5012 \times 10^{-4}</td>
<td>( 1 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.0013</td>
<td>3.4362 \times 10^{-5}</td>
<td>( 1 \times 10^{-8} )</td>
</tr>
</tbody>
</table>

**Table 2.** Comparison of the numerical solutions obtained by the present method and those obtained by LY at \( t = 10, \alpha = 0.7 \) for various values of \( h \).

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0018</td>
<td>0.0036</td>
<td>( 7 \times 10^{-5} )</td>
</tr>
<tr>
<td>0.05</td>
<td>7.1792 \times 10^{-4}</td>
<td>0.0011</td>
<td>( 1 \times 10^{-5} )</td>
</tr>
<tr>
<td>0.025</td>
<td>2.7651 \times 10^{-4}</td>
<td>3.402 \times 10^{-4}</td>
<td>( 2 \times 10^{-6} )</td>
</tr>
<tr>
<td>0.0125</td>
<td>1.0326 \times 10^{-4}</td>
<td>1.0651 \times 10^{-4}</td>
<td>( 5 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.00625</td>
<td>3.7853 \times 10^{-5}</td>
<td>3.4101 \times 10^{-5}</td>
<td>( 1 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.003125</td>
<td>1.3714 \times 10^{-5}</td>
<td>1.1161 \times 10^{-5}</td>
<td>( 2 \times 10^{-8} )</td>
</tr>
</tbody>
</table>

**Table 3.** Comparison of the numerical solutions obtained by the present method and those obtained by LY at \( t = 1, \alpha = 0.5 \) for various values of \( h \).

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0066</td>
<td>0.9 \times 10^{-4}</td>
<td>( 3 \times 10^{-5} )</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0031</td>
<td>0.1 \times 10^{-4}</td>
<td>( 5 \times 10^{-6} )</td>
</tr>
<tr>
<td>0.025</td>
<td>0.0014</td>
<td>0.6 \times 10^{-4}</td>
<td>( 9 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.5 \times 10^{-3}</td>
<td>0.3 \times 10^{-4}</td>
<td>( 1 \times 10^{-7} )</td>
</tr>
<tr>
<td>0.00625</td>
<td>0.2 \times 10^{-3}</td>
<td>0.1 \times 10^{-4}</td>
<td>( 2 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.003125</td>
<td>0.1 \times 10^{-3}</td>
<td>0.6 \times 10^{-5}</td>
<td>( 5 \times 10^{-9} )</td>
</tr>
</tbody>
</table>

**Table 4.** Comparison of the numerical solutions obtained by the present method and the ones obtained by LY with \( t = 10, \alpha = 0.3 \) for various values of \( h \).

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0021</td>
<td>0.0161</td>
<td>( 9 \times 10^{-6} )</td>
</tr>
<tr>
<td>0.05</td>
<td>4.5357 \times 10^{-4}</td>
<td>0.0059</td>
<td>( 1 \times 10^{-6} )</td>
</tr>
<tr>
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<td>0.0021</td>
<td>( 3 \times 10^{-7} )</td>
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<tr>
<td>0.0125</td>
<td>1.1965 \times 10^{-5}</td>
<td>7.5212 \times 10^{-4}</td>
<td>( 5 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.00625</td>
<td>3.3313 \times 10^{-5}</td>
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<td>( 1 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.003125</td>
<td>1.2418 \times 10^{-6}</td>
<td>9.8093 \times 10^{-5}</td>
<td>( 2 \times 10^{-9} )</td>
</tr>
</tbody>
</table>
Example 5.3. Consider

\[ cD_0^\alpha x(t) - t \cos t + x(t) - (1 + t) \sin t = 0 \quad 0 < \alpha \leq 1, x(0) = 1. \]

For \( \alpha = 1 \), the exact solution is \( x(t) = e^{-t} + t \sin t \) [19]. The numerical solutions are computed in Table 5 for \( \alpha = 0.3, 0.5, 0.7, 1 \) at \( t = 1 \). Also we list the absolute errors (AE) for \( \alpha = 1 \). In Fig. 1, we plot the numerical solutions at \( h = 0.05 \) for various values of \( \alpha = 0.5, 0.7, 0.9 \) and \( \alpha = 1 \).

![Fig. 1](image)

**Table 5.** Values of \( x(t) \) in example 3 at \( t = 1 \) for \( \alpha = 0.3, 0.5, 0.7, 1 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \alpha = 0.3 )</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 0.7 )</th>
<th>( \alpha = 1 )</th>
<th>( AE(\alpha = 1) )</th>
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</thead>
<tbody>
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<td>0.1</td>
<td>1.5289</td>
<td>1.4411</td>
<td>1.3527</td>
<td>1.2092</td>
<td>( 2 \times 10^{-4} )</td>
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<td>0.05</td>
<td>1.5248</td>
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</tr>
<tr>
<td>0.025</td>
<td>1.5217</td>
<td>1.4364</td>
<td>1.3502</td>
<td>1.2093</td>
<td>( 2 \times 10^{-6} )</td>
</tr>
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<td>0.0125</td>
<td>1.5192</td>
<td>1.4352</td>
<td>1.3498</td>
<td>1.2094</td>
<td>( 3 \times 10^{-7} )</td>
</tr>
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<td>0.00625</td>
<td>1.5172</td>
<td>1.4345</td>
<td>1.3496</td>
<td>1.2094</td>
<td>( 3 \times 10^{-8} )</td>
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<td>0.003125</td>
<td>1.5155</td>
<td>1.4339</td>
<td>1.3494</td>
<td>1.2094</td>
<td>( 5 \times 10^{-9} )</td>
</tr>
</tbody>
</table>
Example 5.4. Consider the fractional differential equation [19]

\[ \mathcal{D}_0^\alpha t x(t) = -2tx^2(t) + x(t)(1 + t^2) - 1, \quad 0 < \alpha \leq 1, x(0) = 1. \]

For \( \alpha = 1 \), the exact solution is \( x(t) = \frac{1}{1+t^2} \). The numerical solutions computed at \( t = 1 \) for \( \alpha = 0.3, 0.5, 0.7, 1 \) are given in Table 5. Also we list the AE at \( \alpha = 1 \). In Fig. 2, we plot the numerical solutions at \( h = 0.05 \) for \( \alpha = 0.5, 0.7, 0.9 \) and \( \alpha = 1 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \alpha = 0.3 )</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 0.7 )</th>
<th>( \alpha = 1 )</th>
<th>( AE(\alpha = 1) )</th>
</tr>
</thead>
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<td>0.1</td>
<td>0.3902</td>
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<td>4.999</td>
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<td>0.4391</td>
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<tr>
<td>0.0125</td>
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<td>0.4085</td>
<td>0.4391</td>
<td>0.5</td>
<td>( 3 \times 10^{-7} )</td>
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<tr>
<td>0.00625</td>
<td>0.3962</td>
<td>0.4085</td>
<td>0.4391</td>
<td>0.5</td>
<td>( 3 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.003125</td>
<td>0.3962</td>
<td>0.4085</td>
<td>0.4391</td>
<td>0.5</td>
<td>( 5 \times 10^{-9} )</td>
</tr>
</tbody>
</table>

Fig. 2 The numerical solutions at \( h = 0.02 \) for \( \alpha = 0.3, 0.5, 0.7 \) and \( \alpha = 1 \).
Example 5.5. We consider [19]

\[
\begin{align*}
{^cD_0^\alpha}x_1(t) - tx_2(t) + t^2 \cos(t) + x_1(t) - (1 + t)x_2(t) + (t^2 + 2t) \sin(t) &= 0, \\
{^cD_0^\alpha}x_2(t) - \cos(t) - x_2(t) + (t - 1) \sin(t) &= 0, \quad x_1(0) = x_2(0) = 1.
\end{align*}
\]

When \(\alpha_1 = \alpha_2 = 1\), the exact solution is \(x_1(t) = e^{-t} + te^t, x_2(t) = e^t + t \sin t\). In Table 7, we list the numerical solutions for \(x_1(t)\) and \(x_2(t)\) at \(t = 1\) for \(\alpha_1 = 0.5, \alpha_2 = 0.7\). Also we list the AE of \(x_1(t)\) and \(x_2(t)\) at \(\alpha_1 = \alpha_2 = 1\). In Figures 3 and 4, we plot the numerical solutions and the exact solutions \(x_1\) and \(x_2\) at \(h = 0.1, \alpha = 1\).

![Figure 3](image.png)

**Fig. 3** The numerical and the exact solution of \(x_1\) at \(h = 0.1\) for \(\alpha = 1\).
Fig. 4  The numerical and the exact solution of $x_2$ at $h = 0.1$ for $\alpha = 1$.

**Table 7.** Values of $x_1(t), x_2(t)$ in example 5 at $t = 1$ for $\alpha_1 = 0.5, \alpha_2 = 0.7$ and $AE$ at $\alpha_1 = \alpha_2 = 1$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$AE(x_1)$</th>
<th>$AE(x_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>5.7864</td>
<td>5.0429</td>
<td>$8 \times 10^{-4}$</td>
<td>$2 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.05</td>
<td>5.7921</td>
<td>5.0458</td>
<td>$7 \times 10^{-5}$</td>
<td>$2 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.025</td>
<td>5.7961</td>
<td>5.0484</td>
<td>$8 \times 10^{-6}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.0125</td>
<td>5.7864</td>
<td>5.0429</td>
<td>$8 \times 10^{-4}$</td>
<td>$4 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.00625</td>
<td>5.8011</td>
<td>5.0521</td>
<td>$1 \times 10^{-7}$</td>
<td>$6 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.003125</td>
<td>5.8025</td>
<td>5.0532</td>
<td>$1 \times 10^{-8}$</td>
<td>$7 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

6 Conclusions

In this paper, we have developed a new numerical method for solving fractional differential equations. As a first step, the fractional order differential equations are converted into the integer order differential equations. Then we apply the waveform relaxation method to solve the resulting equations. The efficiency and accuracy of the proposed method has been demonstrated by applying it on five typical examples. It is found that the approximate solutions produced by our method are in complete
agreement with the corresponding exact solutions. Moreover, in view of its simplicity, our method is applicable to a wide class of applied problems.

References